Milnor's Exotic Spheres

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Abstract

This manuscript serves as an exposition of a mathematical structure discovered by John Milnor in 1956: the *exotic sphere*. The construction served as a counterexample to the differentiable Poincaré conjecture, which was well-accepted in its time. The discovery of exotic spheres led to a revolution of approaches in the relatively young field of differential geometry, as well as a race to discover more exotic structures in other dimensions. In 4 dimensions, the problem still remains open.

We offer a comprehensive breakdown of Milnor's argument proving the existence of exotic spheres: manifolds that are homeomorphic but not diffeomorphic to the 7-sphere. Aimed at an undergraduate or early graduate level, we first cover the tools and concepts needed to understand Milnor's construction, with particular focus on fibre bundles, cohomology theory and characteristic classes. We construct candidates for exotic spheres via clutching maps, producing S^3 bundles over S^4. An application of a result of Morse Theory allows us to conclude that some of these clutching maps give rise to topological 7-spheres. We then develop an identity which must be satisfied by the candidates in order to be diffeomorphic to the 7-sphere. Non-solutions to the identity correspond to Milnor's exotic spheres, proving their existence.

Acknowledgements

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We would also like to extend our thanks to the late Dr. Andrew Ranicki, for his wonderful collection of online resources on exotic spheres.

Contents

Index of Notation

General

S^n	standard n-sphere
$\mathbb{H}$	division algebra of quaternions
$\cong$ (topology)	homeomorphic
$\cong$ (groups)	isomorphic

Homotopy

$\simeq$	homotopy equivalence
$\pi_n(X)$	n^{th} homotopy group of X
$\pi_n(X, A)$	n^{th} homotopy group of X rel A
$X \vee Y$	one-point compactification of X and Y
$f \vee g$	applied to $X \vee Y : f(X) \vee g(Y)$

Homology

$\triangle^n$	n-simplex
$C_n(X)$	n^{th} chain group of X
∂_n	n^{th} boundary map $C_n(X) \to C_{n-1}(X)$
$H_n(X)$	n^{th} homology group of X
$C_n(X, A)$	n^{th} chain group of X rel A
$H_n(X, A)$	n^{th} homology group of X rel A
$H_*(X)$	homology ring of X

Cohomology

$C^n(X; G)$	n^{th} cochain group of X with coefficients in G
δ_n	n^{th} coboundary map $C^n(X; G) \to C^{n+1}(X; G)$
$H^n(X; G)$	n^{th} cohomology group of X with coefficients in G
$C^n(X, A; G)$	n^{th} cochain group of X rel A with coefficients in G
$H^n(X, A; G)$	n^{th} cohomology group of X rel A with coefficients in G

$H^*(X;G)$	cohomology ring of X with coefficients in G
$\smile$	cup product

Characteristic Classes

e	Euler class
c	Chern class
p	Pontryagin class

For the Proof of Exotic Spheres

$SO(4)$	Group of rotations in 4-dimensional Euclidean space
E_φ	vector bundle over S^n constructed via clutching map $\varphi : S^{n-1} \to GL_k(\mathbb{R})$
$\xi_{i,j}$	vector bundle over S^4 with clutching map indexed $f_{i,j} : S^3 \to SO(4)$
$E_{i,j}$	sphere bundle associated to $\xi_{i,j}$
$\sigma(X)$	signature of X
TX	tangent bundle over X
$\lambda(X)$	Milnor's invariant of X
$\mathbb{HP}^n$	quaternionic projective space
$\mathbb{HP}^1$	quaternionic projective line
$\gamma_{\mathbb{H}}$	tautological line bundle of $\mathbb{HP}^1$

Chapter 1

Introduction

A common and important aim of mathematics is to classify structures. In what ways are two objects the same, and in what ways do they differ? In the history of mathematics, many famous means of classification have been discovered, such as the classfication of finite simple groups, a result that took over 50 years and contribution from many mathematicians to prove, and the classification of semisimple algebras by the Artin-Wedderburn theorem.

In the past century, attempts to classify objects in topology and differential geometry have also been fruitful. In particular, we have had some success in classifying manifolds that share certain properties with the n-dimensional sphere, S^n. For low values of n, we naturally have good intuition for what the n-sphere looks like. If we take $S^n = \{x \in \mathbb{R}^{n+1} \ : \ \|x\| = 1\}$, the set of points in $\mathbb{R}^{n+1}$ a distance 1 from the origin, then we can easily visualise the sphere embedded in Euclidean space, as shown in Figure 1.

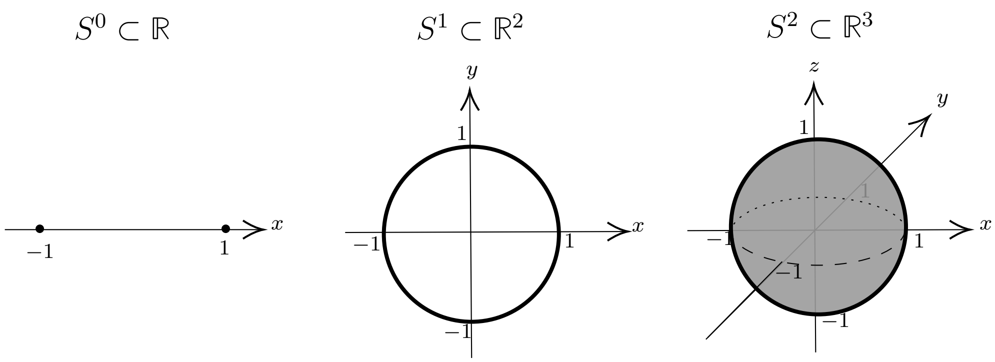


Figure 1.0.1: Low dimensional spheres embedded in $\mathbb{R}^n$.

A major area of research in differential topology involves understanding the relationships between topological and differentiable equivalences on manifolds. Related to spheres, two important questions have been asked:

1. There are several notions of a manifold being topologically the same as S^n. Are these notions equivalent?
2. If a manifold is topologically the same as S^n, is it also necessarily the same as S^n in the differentiable sense?

These questions can be formalised using concepts and techniques available in the 20^{th} century, in particular the notions of *homotopy equivalence*, *homeomorphism*, and *diffeomorphism*. In the case of spheres, we have *homotopy spheres*: manifolds that are homotopy equivalent to the n-sphere. We also have *topological spheres* which are manifolds that are homeomorphic to the n-sphere. The *topological* Poincaré Hypothesis asserts that every closed topological manifold with the homotopy type of an n-sphere is a topological sphere. The proof of this hypothesis by [Sma61], [New66], and [Per02] showed that the notions of homotopy spheres and topological spheres are equivalent in all dimensions, answering the first question. The *differentiable* Poincaré Hypothesis would assert that every closed, smooth manifold with the homotopy type of an n-sphere is diffeomorphic to the n-sphere. Notably, the differentiable case is not as definite as the topological case. As demonstrated by the prominent 20^{th}-century mathematician John Milnor, the differentiable Poincaré Hypothesis is not true in all dimensions [Mil11b]. In dimension 4, the conjecture remains open.

To illustrate how some of these notions differ, consider the following homeomorphism that is not a diffeomorphism: $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$. This is clearly a homeomorphism (it is continuous with a continuous inverse), but it is not a diffeomorphism since its inverse $f^{-1}(x) = x^{\frac{1}{3}}$ is not differentiable at 0. This example, however, does not illustrate that there is *no* possible diffeomorphism $\mathbb{R} \to \mathbb{R}$. It was believed in the early 20^{th} century that every homeomorphism could be "smoothed" to give a diffeomorphism. In other words, any topological equivalence between objects would mean that they also shared a differentiable structure. In 1953 whilst studying $2n$-dimensional manifolds that are $(n-1)$-connected [Mil07], Milnor made the discovery of *exotic spheres*, providing a counterexample to this prevalent belief. He gave the first example of a manifold that is homeomorphic to the standard 7-sphere but is not diffeomorphic to it [Mil56].

Milnor constructed these exotic spheres as the total spaces of some 3-sphere bundles over the 4-sphere. He then showed that they are homeomorphic to S^7 by using *Morse theory*, and that they are not diffeomorphic to S^7 by constructing an invariant up to diffeomorphism of 7-manifolds using *Hirzebruch's Signature Theorem*.

In this project we will give an exposition of Milnor's construction of exotic 7-spheres. This requires an extensive amount of background material, which is presented in Chapter 2. We introduce fibre bundles, the theories of homotopy, homology, and cohomology, as well as characteristic classes.

In Chapter 3 we give the construction of the candidate manifolds for exotic spheres, and conclude our exposition by proving that some are homeomorphic but not diffeomorphic to S^7.

Chapter 2

Preliminary Reading

Understanding Milnor's argument proving the existence of exotic spheres requires a great deal of background that an undergraduate or early graduate student may not have already. To compensate, this section serves as a rapid introduction to all basic concepts needed. Proofs of results have generally been omitted, though we have clearly included references to where the reader can find them.

We assume that the reader has had a thorough introduction to manifolds and some basic algebraic topology. Specifically, we will assume a complete understanding of the contents of the University of Edinburgh's Differentiable Manifolds course [FO20], as well as its Algebraic Topology course [PR21]. We also assume that the reader is aware of some rudimentary category theory, in particular the notions of covariant and contravariant functors between categories. For readers unfamiliar with these concepts, a brief glance at Chapter 1 of Tom Leinster's *Basic Category Theory* [Lei14] will suffice.

We begin our background by introducing bundles, in particular fibre bundles and some important related constructions. We then go on to examine homotopy, homology and cohomology theory. The last section of the background focuses on characteristic classes, which ties all aforementioned concepts together.

2.1 Bundles

Definition 2.1.1. A bundle over an object X in a category $\mathcal{C}$ is just an object $E \in \mathcal{C}$ and a morphism π

$$\begin{tikzcd} E \arrow[d, "\pi"] \\ X \end{tikzcd}$$

In other words, a **bundle** is defined to be a triple, comprising the **base space** X, the **total space** E, and a map between them $\pi : E \to X$.

Let X be a smooth manifold. We assume the reader has familiarity with the notion of a rank-m, real vector bundle over X, say $\mathbb{R}^m \to E \to X$. The vector bundle is in fact a special case of a more general construction known as a **fibre bundle**.

To define fibre bundles, or more a specific type of fibre bundle known as a principal bundle, we will first briefly introduce the notion of a topological group and a right action. This section draws on Husemoller's Fibre Bundles [Hus66].

Definition 2.1.2. We say that G is a **topological group** if G is a set with a group structure as well as a topological structure, satisfying the condition that the functions $(u, v) \mapsto uv$ and $u \mapsto u^{-1}$ are both continuous maps from $G \times G \to G$ and $G \to G$ respectively.

Definition 2.1.3. Let G be a topological group, and X a space. We say that X is a **right G-space** if there exists a map $X \times G \to X$ given by $(x, g) \mapsto xg$, and satisfying

1. (Associativity) For all $x \in X$ and $g, h \in G$, $x(gh) = (xg)h$.
2. (Identity) For all $x \in X$, $x1 = x$, where 1 is the identity in G.

Any map $X \times G \to X$ satisfying the above is then known as a (right) action of G (on X).

These definitions may sound abstract, but they describe spaces we already know. For example, $\mathbb{R}^4$ is a left $GL(4, \mathbb{R})$-space, as well as a left (or equally right) $(\mathbb{R} - \{0\})$-space. We will denote the space of G-spaces by $\mathbf{sp}_G$. We say that a map $f : X \to Y$ from one G-space to another is a **G-morphism** so long as it satisfies $f(xg) = f(x)g$ for all $x \in X$, $g \in G$.

Definition 2.1.4 (G-Principal Bundle)**.** For a (topological) group G (known as the **structure group**) and some space X, a **G-principal bundle** over a space X is a space E (more specifically a bundle $E \to X$) with a free and transitive [1] action of G such that the space $E \to X$ is isomorphic to the quotient map $E \to E/G$ [2]. Since the action of G is free and transitive, each fiber of $E \to X$ looks like G when we pick a base point. We denote this by

$$\begin{array}{ccc} G & \hookrightarrow & E \\ & & \downarrow \\ & & Y \end{array}$$

[1] A free action $X \times G \to X$ is one where $xg = x$ implies g is the identity; a transitive action is one where for every $x, y \in X$, there exists a g such that $xg = y$.

[2] By convention this action is taken to be a right action.

Principal bundles possess the useful property that they pull-back. Suppose we have a map $f : X \to Y$ and a principal bundle over Y

$$\begin{tikzcd} G \arrow[r, hook] & E \arrow[d] \\ & Y \end{tikzcd}$$

then we get out of it a principal bundle over X

$$\begin{tikzcd} G \arrow[r, hook] & f^*E \arrow[d] \\ & X \end{tikzcd}$$

But what is even more interesting is that from this fact we have a very neat way of describing all G-principal bundles:

Theorem 2.1.5. Given a topological group G, there exists a G-principal bundle

$$\begin{tikzcd} G \arrow[r, hook] & EG \arrow[d] \\ & BG \end{tikzcd}$$

where BG is called the **classifying space**, such that EG is contractible. This bundle is unique up to homotopy, a classifying property of a topological space which is discussed in later sections. This bundle is known as the G**-universal principal bundle**. G-principal bundles are part of a bigger class of bundles, known as **fibre bundles**.

Definition 2.1.6. (fibre bundle) A fibre bundle, denoted by

$$\begin{tikzcd} F \arrow[r, hook] & E \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

is a collection of objects isomorphic to F which are parametrized by points in the base space X. Locally, the total space looks like an ordinary cartesian product $X \times F$. The triple (E, X, π) satisfies the following conditions

1. The base space X is covered by open sets $\{U_\alpha\}$ on which $\pi^{-1}(U_\alpha)$ is homeomorphic to $U_\alpha \times F$

2. The local trivialisations $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ are such that the following diagram commutes

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[r, "\phi_\alpha"] \arrow[d, "\pi"'] & U_\alpha \times F \arrow[dl, "\mathrm{pr}_1"] \\ U_\alpha & \end{tikzcd}$$

The trivial bundle $E = X \times F$ is the simplest example of a fibre bundle where π is given by projection onto the first coordinate, denoted pr_1.

Example 2.1.7. A vector bundle

$$\begin{array}{c} E \\ \downarrow \; \pi \\ X \end{array}$$

is a familiar example of a fibre bundle. The fibres are vector spaces V_x for $x \in X$

Definition 2.1.8. (sphere bundle) A **sphere bundle** is a fibre bundle in which all the fibres are $n-$dimensional spheres S^n.

Sphere bundles will be the most important type of fibre bundle we will encounter. In section 3 we will show how exotic spheres can be constructed as total spaces of sphere bundles.

2.1.1 Hopf Fibrations

The Hopf fibration is a very famous example of a sphere bundle in which we fibre S^3 over S^2 such that each fibre is homeomorphic to S^1.

$$\begin{tikzcd} S^1 \arrow[r, hook] & S^3 \arrow[d, "\pi"] \\ & S^2 \end{tikzcd}$$

Since S^2 is homeomorphic to the complex projective line $\mathbb{CP}^1$, we can define a map $\pi : S^3 \to S^2 \cong \mathbb{CP}^1$ by $\pi(x^1, x^2) = [x^1 : x^2]$. Letting $x^1 = 1$, $x^2 = 0$ we see that $\pi^{-1}([1:0]) = (z, 0) \in S^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$. So we have that $\pi^{-1}([1:0]) = \{z \in \mathbb{C} : |z|^2 = 1\} \cong S^1$.

2.1.2 Quaternions and Quaternionic Hopf Fibrations

The construction can be generalised to other division algebras, namely $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. We expect that the latter two division algebras, the quaternions and octonions, are unfamiliar to the reader. We will not discuss the octonions, but will describe the case of the quaternions in depth.

Definition 2.1.9. The **quaternions**, $\mathbb{H}$ are the division algebra

$$\mathbb{H} := \{a + bi + cj + dk | a, b, c, d \in \mathbb{R}^4\}$$

where i, j, k satisfy

$$i^2 = j^2 = k^2 = ijk = -1,$$

$$
\begin{aligned}
ij &= -ji = k, \\
jk &= -kj = i, \\
ki &= -ik = j.
\end{aligned}
$$

It is simple to see that this is indeed a division algebra. But we can also consider $\mathbb{H}$ as a 4-dimensional real vector space, isomorphic to $\mathbb{R}^4$.

We say that an element $q \in \mathbb{H}$ is a unit quaternion if q satisfies $||q|| := \sqrt{q\bar{q}} = 1$, where the **conjugate** $\bar{q}$ is given by $\bar{q} = a - bi - cj - dk$. Analogously to the complex numbers, we can identify the set of unit quaternions with S^3. We can also see that, in a similar way to how we described S^3 in the above complex Hopf fibration, $S^7 = \{(q, p) \in \mathbb{H} \times \mathbb{H} : |q|^2 + |p|^2 = 1\}$.

We can now define the quaternionic Hopf fibration by fibring S^7 over S^4 to get a fibre bundle

$$
\begin{tikzcd}
S^3 \arrow[r, hook] & S^7 \arrow[d, "\pi"] \\
& S^4
\end{tikzcd}
$$

where the map $\pi : S^7 \to S^4$ sends $(q_1, q_2) \in S^7$ to $[q_1 : q_2] \in \mathbb{HP}^1 \cong S^4$ (as proved in [Wal03] page 246). To see that the fibres are homeomorphic to S^3, consider that from the above characterisation of S^7 and the fact that we can identify the set of unit quaternions with S^3, $\pi^{-1}([1 : 0]) = \{(q, 0) \in \mathbb{H} : |q|^2 = 1\} \cong S^3$.

We will need several more facts for our later calculations involving quaternions.

Definition 2.1.10. We define the **real part** of a quaternion $q = a + bi + cj + dk$ to be $R(q) := a$.

Definition 2.1.11. The **product** of two quaternions $a_1+b_1i+c_1j+d_1k$ and $a_2+b_2i+c_2j+d_2k$ is the quaternion

$$
\begin{aligned}
a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2 + (a_1b_2 + b_1a_2 + c_1d + 2 - d_1c_2)i \\
+ (a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2)j + (a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2)k.
\end{aligned}
$$

Notice that the product defined above is associative but not commutative.

Definition 2.1.12. The **reciprocal** of a quaternion q is defined to be

$$
q^{-1} := \frac{\overline{q}}{\|q\|^2}.
$$

By brute force expansion, we have the following properties:

Proposition 2.1.13. For any quaternion q_1, q_2:

1. $\overline{q_1 q_2} = \overline{q_2} \cdot \overline{q_1}$;
2. $\|q_1 q_2\| = \|q_1\|\|q_2\|$;
3. $q_1^{-1} q_2^{-1} = \frac{\overline{q_2 q_1}}{\|q_1\|^2 \|q_2\|^2}$.

Proposition 2.1.14. For any unit quaternion q_1 and any quaternion q_2,

$$R(q_1^{-1} q_2 q_1) = R(q_2).$$

2.2 Homotopy

A foundation in homotopy, homology and cohomology theory is essential to understand Milnor's paper. Our explanations will rely heavily on Alan Hatcher's *Algebraic Topology* [Hat02], though we will focus only on what we need in order to understand exotic spheres. Note well that in any instance of the word "map", we mean "continuous function".

Definition 2.2.1. Suppose we have two topological spaces X and Y, and denote $I := [0, 1]$. A **homotopy** from X to Y is a family of maps $f_t : X \to Y$ indexed by $t \in I$, such that the associated map $F : X \times I \to Y$ given by $F(x, t) = f_t(x)$ is continuous.

Example 2.2.2. One example of a homotopy between two spaces is a **deformation retraction**. Suppose we have a topological space X and a subspace $A \subset X$. Then a deformation retraction onto A is any family of maps $f_t : X \to A$ such that $f_0 = \mathbb{1}$, $f_1(X) = A$ and $f_t|_A = \mathbb{1}$, where $\mathbb{1}$ is the identity map.

In general, any homotopy $f_t : X \to Y$ whose restriction to a certain subspace $A \subset X$ is independent of t is called a **homotopy relative to** A , written homotopy rel A.

We also have a notion of homotopy between continuous functions. Suppose f and g are two functions $X \to Y$. Then f and g are said to be **homotopic**, written $f \simeq g$, if there exists a homotopy $h_t : X \to Y$ such that $h_0(x) = f(x)$ and $h_1(x) = g(x)$.

Homotopy is a useful tool to classify topological spaces. A map $f : X \to Y$ is said to be a **homotopy equivalence** if there exists another map $g : Y \to X$ such that $fg \simeq \mathbb{1}$ and $gf \simeq \mathbb{1}$ (by fg we mean $f \circ g$). We say that the spaces X and Y are then **homotopy equivalent** or have the same homotopy type, and denote this by $X \simeq Y$. In layman's terms, we say that $X \simeq Y$ if we can continuously deform, expand and/or shrink X to form Y.

Some easy to visualize examples of homotopy equivalence are capital letters of the alphabet. Considering each as its own topological space equipped with the usual topology, letters such as A, D, O, and P are all homotopy equivalent, each continuously deformable into a circle. The

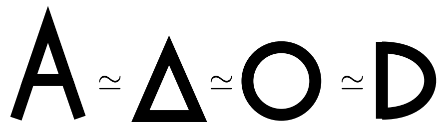


Figure 2.2.1: Visualising the homotopy equivalence of letters of the alphabet.

letter B is rather lonely, being only homotopy equivalent to itself (there are no other capital letters in the English language with two "holes"). Many letters such as C, E, F, G, H, I, J... are **contractible** - that is, homotopy equivalent to a point.

We know a lot of homotopic equivalent spaces already - homeomorphic ones. If two spaces X and Y are homeomorphic with homeomorphism $f : X \to Y$, then by definition f is a continuous map between X and Y with continuous inverse f^{-1}. Since $ff^{-1} = f^{-1}f = \mathbb{1}$, the weaker statement that $ff^{-1} \simeq \mathbb{1}$ and $f^{-1}f \simeq \mathbb{1}$ trivially holds.

We can formalise homotopy equivalence further by constructing so-called **homotopy groups**.

Lemma 2.2.3. Homotopy equivalence defines an equivalence relation on the space of all topological spaces.

Proof. Of course any topological space is trivially homeomorphic to itself, and so must also be homotopy equivalent to itself. This proves reflexivity. Symmetry follows simply by the fact that the definition of homotopy equivalence is symmetric. To prove transitivity, Suppose X, Y, and Z are topological spaces and $X \simeq Y$,$Y \simeq Z$. Then we have homotopy equivalences $f : X \to Y$ and $j : Y \to Z$ with homotopy inverses $g : Y \to X$ and $k : Z \to Y$ respectively.

It is enough to prove that if $r_1, r_2 : X \to Y$ and $s_1, s_2 : Y \to Z$ are smooth maps satisfying $r_1 \simeq r_2$ and $s_1 \simeq s_2$, then $s_1 \circ r_1 \simeq s_2 \circ r_2$. Notice that if this is true, then immediately we have that $fj : X \to Z$ is a homotopy equivalence with homotopy inverse $gk : Z \to X$:

$$(gk)(jf) = g(kj)f \simeq g\mathbb{1}_Y f = gf \simeq \mathbb{1}_X$$

and

$$(jf)(gk) = j(fg)k \simeq j\mathbb{1}_Y k = jk \simeq \mathbb{1}_Z.$$

To prove this result then, notice that if $r_1 \simeq r_2$ and $s_1 \simeq s_2$, then there exist homotopies $\alpha : X \times I \to Y$ with $\alpha(x, 0) = r_1(x)$, $\alpha(x, 1) = r_2(x)$ and $\beta : Y \times I \to Z$ with $\beta(y, 0) = s_1(y)$, $\beta(y, 1) = s_2(y)$.

Composing these homotopies gives us another homotopy:

$$h(x, t) := \beta(\alpha(x, t), t).$$

We can see that it is continuous, but also $h(x, 0) = s_1 r_1(x)$ and $h(x, 1) = s_2 r_2(x)$. □

This allows us to very quickly identify when topological spaces are homotopy equivalent. For example, the Möbius band is homotopy equivalent to the normal band, since they are both homotopy equivalent to a circle.

A more obvious fact is the following:

Lemma 2.2.4. Homotopy defines an equivalence relation on the space of all maps between two topological spaces.

Proof. Let X and Y be two topological spaces, and $f, g, h : X \to Y$ be maps between them. Then $f \simeq f$ by the constant homotopy $p : X \times I \to Y$ which maps $(x,t) \mapsto f(x)$. If $f \simeq g$, then there exists a homotopy $q : X \times I \to Y$, $q(x,t) \mapsto q_t(x)$ with $q_0(x) = f(x)$ and $q_1(x) \to g(x)$. We can easily reverse this homotopy by sending $-q : (x,t) \mapsto q_{1-t}(x)$, so that $-q_0 = g(x)$ and $-q_1 = f(x)$ and therefore $g \simeq f$. Finally, if $r : f \simeq g : X \to Y$ and $s : g \simeq h : X \to Y$, then we can construct a homotopy via the concatenation

$$r \cdot s : X \times I \to Y, \quad (x,t) \mapsto \begin{cases} r(x, 2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ s(x, 2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

so that $r \cdot s$ is a homotopy $f \simeq h$. $\square$

It is this equivalence relation that allows us to construct a group from a topological space: a **homotopy group**. Let I^n denote the n-dimensional cube $I^n = [0,1]^n$, and ∂I^n denote its boundary. Suppose X is some space, and choose a base point x_0 in X. We define $\pi_n(X, x_0)$ to be the set of homotopy equivalence classes of maps $f : I^n \to X$ which satisfy the requirement that $f(\partial I^n) = x_0$.

In the case $n = 0$, we take I^0 to be a single point with empty boundary $\partial I^0 = \emptyset$, and thus $\pi_0(X, x_0)$ is simply the set of path-components of X. That is, the set of equivalence classes from the relation identifying path-connected points in X.

This set of homotopy equivalence classes forms a group under the composition action

$$[f] + [g] := [(f+g)(t_1, t_2, ..., t_n)] := \left[\begin{cases} f(2t_1, t_2, ..., t_n) & t_1 \in [0, \frac{1}{2}] \\ g(2t_1 - 1, t_2, ..., t_n) & t_1 \in [\frac{1}{2}, 1]. \end{cases} \right]$$

Though we choose a base point to define these groups, on path-connected components of X this choice is irrelevant; different choices of base points result in isomorphic groups. Therefore, assuming path-connectedness of our space, we can talk about *the* homotopy groups of X, $\pi_n(X)$.

Remark 2.2.5. There is an equivalent way of defining homotopy groups using homotopy equivalent maps from the n-sphere S^n to our space X, rather than from the n-cube, I^n. This is due to the fact that since I^n is homeomorphic to the n-disc D^n, and therefore the quotient $I^n/\partial I^n$ satisfies

$$I^n/\partial I^n \cong D^n/\partial D^n \cong D^n/S^{n-1} \cong S^n.$$

This is easy to visualize for I^2, as depicted in Figure 2.2.2.

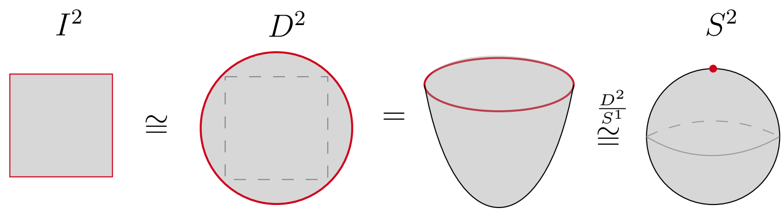


Figure 2.2.2: Visualising homeomorphism $I^2/\partial I^2 \cong S^2$.

It follows that homotopy equivalent maps from $I^n \to X$ which send ∂I^n to x_0 can be equivalently thought of as homotopy equivalent maps from $S^n \to X$ which send $\partial I^n/\partial^n =: s_0 \to x_0$ for a base point $s_0 \in S^n$. We can then interpret the composition action $f+g$ of the elements in $\pi(X, x_0)$ to be the composition $S^n \xrightarrow{c} S^n \vee S^n \xrightarrow{f \vee g} X$, where c is a map that collapses the equator of S^n to a point s_0 which we choose to lie on the equator, [3] and $f \vee g$ is a map that has f act on the upper S^n and g on the lower S^n.

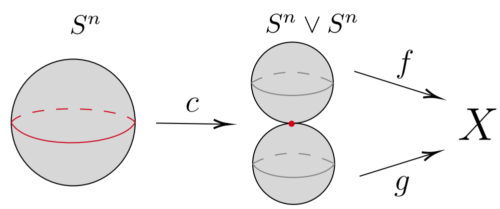


Figure 2.2.3: Constructing Homotopy groups via S^n.

The map $\pi_n : X \to \pi_n(X)$ is actually a covariant functor from the category of topological spaces to the category of groups. If $\phi : X \to Y$ is a map between (path-connected) spaces, then π_n induces a covariant map between homotopy groups $\phi_* : \pi_n(X) \to \pi_n(Y)$ sending $[f] \mapsto [\phi \circ f]$, where $[f]$ is the equivalence class of $f : I^n \to X$ [4] .

Proposition 2.2.6. The first homotopy group $\pi(X)$ of a path-connected space X is the group of equivalence classes of homotopy equivalent loops within X. We give this group a special name, the **fundamental group** of X.

This follows from our sphere characterisation of homotopy groups, considering maps $f : S^1 \to X$.

Example 2.2.7. $\pi_1(D^2) = 0$.

Proof. Let x_0 be any point on D^2 (since D^2 is path-connected, we can choose x_0 arbitrarily). But any loop with initial and termination point x_0 contained in D^2 is contractible:

[3]Recall that $X \vee Y$ denotes the one point union of X and Y; This means we take $X \wedge Y$ to be the disjoint union of X and Y quotiented by the identification of base points $x_0 \in X$ and $y_0 \in Y$.

[4]Note the convention of denoting *covariant* induced maps by a subscript $*$ (as in ϕ_*), and *contravariant* induced maps by a superscript $*$ (as in ϕ^*).

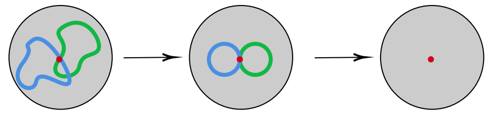


Figure 2.2.4: All loops in D^2 are contractible.

$\square$

Example 2.2.8. $\pi_1(S^1) = \mathbb{Z}$.

Proof. We will give more of an intuition than a rigorous proof. that S^1 can be thought of as an interval $[0, 1]$ with its endpoints identified $0 \sim 1$. We will take our base point to be 0, noting again that this choice is arbitrary since S^1 is path-connected. Any loop that does not circumnavigate the entirety of S^1 is contractible. The identity loop, $f : x \mapsto x$, is not contractible, and so its equivalence class is our first non-trivial element of $\pi_1(S^1)$. Notice though that this loop gives rise to new elements of $\pi_1(S^1)$ via concatenating the loops: running through f once and then again gives a double loop over S^4, which is distinct from the single loop in the same way that the single loop is distinct from the constant map. Constructing elements in this way, it follows that $\pi_1(S^1) \cong \mathbb{Z}$. $\square$

Rather unfortunately, calculating homotopy groups for $n \geq 2$ can quickly become complicated. We do however have some regularity in lower-dimension homotopy groups of S^n:

Proposition 2.2.9.

$$\pi_i(S^n) = \begin{cases} 0 & \text{for } i < n \\ \mathbb{Z} & \text{for } i = n. \end{cases}$$

Proof. For a detailed explanation see [Hat02] p.349 and p. 366. The first case is a consequence of the Cellular Approximation Theorem, and the second a consequence of Hurewicz's Theorem. $\square$

2.2.1 CW Complexes

We will avoid general homotopy, homology and cohomology theory, as it is can become quite abstract and complicated, and in any case we do not need it for our discussion of spheres. Rather, we will restrict our attention to specific types of spaces known as CW Complexes, and in turn consider only cellular homotopy, homology and cohomology.

Definition 2.2.10. A **CW complex** , also known as a cell complex, is a topological space constructed out of subspaces called cells, done in a very specific way. Following the construction in [Hat02] p.5, we can construct a CW complex X in the following way:

1. Begin with a discrete set X^0 of points or "0-cells".

2. Inductively, attach n-cells e^n_α, which can be thought of as n-discs, to the $(n-1)$-skeleton X^{n-1} via mappings $\phi_\alpha \to X^{n-1}$. As a set, $X^n = X^{n-1} \sqcup e^n_\alpha$.

3. Eventually we can stop the process setting $X = X^n$ for some n. [5]

Intuitively, we begin with a space of discrete points, and then successively attach n-discs along their boundaries to the space. This results in rather nice spaces, many of which we already know. The circle is made by attaching a 1-cell to a 0-cell along its boundaries - effectively attaching the ends of a line to a point. More generally, we see that the n-sphere is actually a CW complex, constructed by attaching an n-cell to a 0-cell, as depicted in Figure 2.2.5.

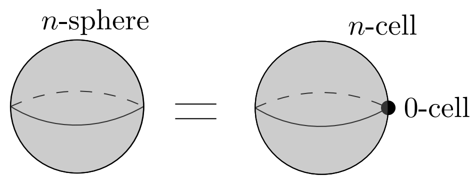


Figure 2.2.5: The n-sphere as a CW-complex.

A fortunate result for our purposes stated in Corollary A.12 of [Hat02] p.529 is the following

Proposition 2.2.11. Closed manifolds are homotopy equivalent to CW complexes.

2.3 Homology

As mentioned, we will restrict our attention to cellular homology. In the build up to this, we will develop the theory for a slightly less complicated homology, **singular homology**. The majority of the content of this section has been adapted from [Hat02].

2.3.1 Singular Homology

Definition 2.3.1. An **n-simplex** is the convex hull of $n+1$ points, $v_0, v_1, ..., v_n$ in $\mathbb{R}^m$ such that they do not lie in a hyperplane of dimension less than n, or equivalently, such that $v_1 - v_0, ..., v_n - v_0$ are linearly independent.

We denote an n-simplex by $[v_0, ...v_n]$. We define the standard n-simplex to be

$$\Delta^n = \{(v_0, ...v_n) \in \mathbb{R}^{n+1} | \sum_i v_i = 1, v_i \geq 0 \ \forall i\}$$

[5]We can actually have *infinite-dimensional* CW complexes, taking the union $X = \cup_n X^n$. We do not consider these spaces.

Definition 2.3.2. A **singular n-simplex** in X is a continuous map $\sigma : \Delta^n \to X$

Definition 2.3.3. The **boundary** of the n-simplex $[v_0, ...v_n]$ is $\sum_i (-1)^i [v_0, ...\hat{v}_i, ...v_n]$ where the hat on a vertex means deleting it from the sequence.

Let $C_n(X)$ be the free abelian group that has the set of singular n-simplices in X as a basis. The groups $C_n(X)$ are called the chain groups of X. Elements of $C_n(X)$ are called **n-chains**. There is a homomorphism $\partial_n : C_n(X) \to C_{n-1}(X)$ taking an n-chain $\sigma \in C_n(X)$ to its boundary $\partial_n(\sigma) = \sum_i (-1)^i \sigma|[v_0, ..., \hat{v}_i, ...v_n]$.

Definition 2.3.4. A **chain complex** is a sequence of abelian groups A_n and homomorphisms $\varphi_n : A_n \to A_{n-1}$ such that $\varphi_n \varphi_{n+1} = 0$

We can check by expanding the definition that $\partial_n \partial_{n+1} = 0$. So in the case above, we have a **singular chain complex**

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

We are now ready to define the n^{th} singular homology group of X.

Definition 2.3.5. The **n^{th} singular homology group of** X, $H_n(X)$ is the quotient group

$$H_n(X) := {}^{\ker \partial_n}\!\big/_{\text{im } \partial_{n+1}}.$$

Elements of $\ker \partial_n$ are called **cycles** and elements of Im ∂_{n+1} are called **boundaries**.

We now present some useful results relating to singular homology taken from [Hat02].

Proposition 2.3.6. Let X be a topological space with path components $\{X_\alpha\}_{\alpha \in A}$. Then

$$H_n(X) \cong \bigoplus_{\alpha \in A} H_n(X_\alpha).$$

Proposition 2.3.7. For a path connected space X, $H_0(X) \cong \mathbb{Z}$. Therefore any space with k path connected components has $H_0(X) \cong \mathbb{Z}^k$.

For proofs of both of these propositions see [Hat02], p.109.

2.3.2 Homotopy Invariance

The aim of this subsection is to prove that if two spaces X, Y are homotopy equivalent then their homology groups are isomorphic.

If we have a map $f : X \to Y$ then this induces a homomorphism $f_\# : C_n(X) \to C_n(Y)$ in the following way: $f_\#$ sends a singular $n-$simplex $\sigma : \Delta^n \to X$ to $f \circ \sigma : \Delta^n \to Y$. We then extend $f_\#$ linearly $f_\#(\sum_i n_i\sigma_i) = \sum_i n_i f \circ \sigma_i$.

An important property of this induced homomorphism is that it obeys $f_\#\partial = \partial f_\#$. To see this we just use the definition of $\partial(\sigma) = \sum_i(-1)^i\sigma|[v_0, ..., \hat{v}_i, ...v_n]$. and calculating

$$f_\#\partial(\sigma) = f_\#(\sum_i(-1)^i\sigma|[v_0, ..., \hat{v}_i, ...v_n]) = \sum_i(-1)^i f \circ \sigma|[v_0, ..., \hat{v}_i, ...v_n] = \partial f_\#(\sigma).$$

This commutativity condition can be summarised by saying that the $f_\#$'s define a **chain map** from the singular chain complex of X to the singular chain complex of Y. Alternatively, in the typical diagrammatic nature of this subject, we can sum this condition up in the following commutative diagram.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & C_n(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & C_{n+1}(X) \arrow[r] \arrow[d, "f_\#"] & \cdots \\
\cdots \arrow[r] & C_{n+1}(Y) \arrow[r, "\partial"] & C_n(Y) \arrow[r, "\partial"] & C_{n-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

$f_\#$ then takes cycles to cycles and boundaries to boundaries. So $f_\#$ induces a homomorphism f_* on the homology groups $f_* : H_n(X) \to H_n(Y)$. This leads to the following proposition from [Hat02].

Proposition 2.3.8. A chain map between chain complexes induces homomorphisms between the homology groups of the two complexes.

We also present without proof the following two results from [Hat02].

Theorem 2.3.9. If two maps $f, g : X \to Y$ are homotopic, then they induce the same homomorphism $f_* = g_* : H_n(X) \to H_n(Y)$.

Corollary 2.3.10. The maps $f_* : H_n(X) \to H_n(Y)$ induced by a homotopy equivalence $f : X \to Y$ are isomorphism for all n.

2.3.3 Relative Homology

Relative homology is a way of exploring how the homology of a space X relates to the homology of one of its subsets A. It can be thought of as homology of X *modulo* A.

Definition 2.3.11. Let X be a space and $A \subset X$ a subset. We define the **relative chain group** $C_n(X, A)$ to be the quotient group $C_n(X) \;/\; C_n(A)$.

Since the chains in A are trivial in $C_n(X,A)$ and the boundary homomorphism $\partial_n : C_n(X) \to C_{n-1}(X)$ maps $C_n(A)$ to $C_{n-1}(A)$ we get an induced boundary homomorphism $\partial_n : C_n(X,A) \to C_{n-1}(X,A)$. This in turn gives us a chain complex

$$\cdots \longrightarrow C_{n+1}(X,A) \xrightarrow{\partial_{n+1}} C_n(X,A) \xrightarrow{\partial_n} C_{n-1}(X,A) \longrightarrow \dots$$

We define the **relative homology groups** $H_n(X,A)$ as usual, using the above chain complex to get

$$H_n(X,A) = {}^{\ker \partial_n}\!\big/_{\mathrm{im}\ \partial_{n+1}}.$$

Definition 2.3.12. A pair (X,A) is said to be a **good pair** if $A \subset X$ is closed in X and A is a deformation retract of some neighborhood in X.

We now discuss some **exact sequences** for relative homology that will be useful later in our project.

Definition 2.3.13. A sequence of group homomorphisms

$$\cdots \longrightarrow A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \longrightarrow \dots$$

Is said to be **exact** if Ker $\alpha_n =$ Im α_{n+1} for all n.

In particular a **short exact sequence** is an exact sequence of the form

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

We have the following short exact sequence

$$0 \longrightarrow C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X,A) \longrightarrow 0$$

Where i denotes the inclusion $C_n(A) \hookrightarrow C_n(X)$ and j is the quotient map $C_n(X) \to C_n(X,A)$. This sequence results in the following long exact sequence of homology groups, the details are ommitted and can be found in [Hat02].

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \to \cdots$$
$$\cdots \to H_0(X,A) \to 0.$$

2.3.4 Cellular Homology

Cellular homology is useful for computations and in fact is actually isomorphic to singular homology. This means that it is independent of the cell structure of the space we are considering.

Lemma 2.3.14 ([Hat02], p.137)**.** Let X be a CW complex, and consider the good pair (X^n, X^{n-1}) $H_k(X^n, X^{n-1})$. Then

1. $H_k(X^n, X^{n-1}) = \begin{cases} 0 & n \neq k \\ \mathbb{Z}^{\text{number of } n\text{-cells}} & n = k \end{cases}$
2. $H_k(X^n) = 0$ if $k > n$
3. the inclusion $i : X^n \hookrightarrow X$ induces an isomorphism $H_k(X^n) \to H_k(X)$ if $k < n$

Definition 2.3.15. The cellular homology of a CW complex X is the homology groups of the chain complex that is the horizontal line in Figure 2.3.1, from Hatcher [Hat02], p.139.

$$
\begin{tikzcd}
 & & & & 0 & & \\
 & & & H_n\left(X^{n+1}\right) \cong H_n(X) \arrow[ur] & & & \\
 & 0 \arrow[dr] & & & & & \\
 & & H_n\left(X^n\right) \arrow[ur] \arrow[dr, "j_n"] & & & & \\
\cdots \arrow[r] & H_{n+1}\left(X^{n+1}, X^n\right) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n\left(X^n, X^{n-1}\right) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"'] & & H_{n-1}\left(X^{n-1}, X^{n-2}\right) \arrow[r] & \cdots \\
 & & & & H_{n-1}\left(X^{n-1}\right) \arrow[ur, "j_{n-1}"'] & & \\
 & & & 0 \arrow[ur] & & &
\end{tikzcd}
$$

Figure 2.3.1: Cellular homology

In this figure, we can see that the boundary maps for cellular homology are defined by $d_{n+1} = j_n \partial_{n+1}$ and $d_n = j_{n-1} \partial_n$. From the definition of a chain complex we must have $d_n d_{n+1} = 0$ but since ∂_n and ∂_{n+1} are boundary maps for singular homology, we get that as a consequence that $d_n d_{n+1} = 0$. The homology groups of this chain complex are called the cellular homology groups of X. We will denote them for now by $H_n^{CW}(X)$.

Theorem 2.3.16 ([Hat02], p.140)**.** $H_n^{CW}(X) \cong H_n(X)$ where $H_n(X)$ is the n^{th} singular homology group of X.

2.4 Cohomology

In the previous section we saw how to construct the homology groups of n-simplexes and Δ-complexes from a chain complex. The idea of cohomology is very similar, but this time we dualize each free abelian group C_n in a general chain complex C by replacing it with its dual **cochain group** $C_n^* = \text{Hom}(C_n, G)$, which is the (abelian) group of homomorphisms $C_n \to G$, where G is a fixed abelian group called the **coefficients** of the cohomology. We usually take $G = \mathbb{Z}$, the set of integers. As a result, the boundary maps $\partial : C_n \to C_{n-1}$ are replaced by their dual **coboundary maps** $\delta : C_{n-1}^* \to C_n^*$. The arrow is reversed since given a homomorphism $\alpha \in C_{n-1}^* : C_{n-1} \to G$, the map ∂ induces a homomorphism $\beta \in C_n^* : C_n \to G$ by $\beta = \alpha \circ \partial$. Also, since $\partial\partial = 0$, we have $\delta\delta = 0$ as well, hence we have created a reversed chain complex, called the **cochain complex**:

$$\cdots \leftarrow C_{n+1}^* \xleftarrow{\delta_n} C_n^* \xleftarrow{\delta_{n-1}} C_{n-1}^* \leftarrow \cdots$$

We then define the $\mathbf{n^{th}}$ **cohomology groups** $H^n(C; G) = \text{Ker }\delta_n / \text{Im }\delta_{n-1}$, similar to how we defined the homology groups.

Remark 2.4.1. Note that $H^n(C; G)$ is generally not isomorphic to $\text{Hom}(H_n(C), G)$, but there does exist a homomorphism $h : H^n(C; G) \to \text{Hom}(H_n(C), G)$ given as follows. A class in $H^n(C; G)$ is represented by a $\psi \in C_n^*$ such that $\delta_n \psi = 0$, which is equivalent to $\psi\, \partial_{n+1} = 0$. Therefore ψ vanishes on $\text{Im }\partial_{n+1}$. By the First Isomorphism Theorem, the restriction $\psi|_{\text{Ker }\partial_n}$ then induces a homomorphism $\psi_0 : \text{Ker }\partial_n / \text{Im }\partial_{n+1} \to G$, which is in $\text{Hom}(H_n(C), G)$. It is easy to check that if $\psi \in \text{Im }\delta_{n-1}$, then the induced homomorphism $\psi_0 = 0$. Thus there is a well-defined quotient map $h : H^n(C; G) \to \text{Hom}(H_n(C), G)$ sending ψ to ψ_0.

2.4.1 Singular Cohomology

Now we narrow down to the case of **singular cohomology**, which is the dual of singular homology. We dualize the singular chain groups $C_n(X)$ to define $C^n(X; G) = \text{Hom}(C_n(X), G)$, the **singular n-cochains with coefficients in G**, where G is again a fixed abelian group. An element $\varphi \in C^n(X; G)$ assigns to each singular n-simplex $\sigma : \Delta^n \to X$ a value $\varphi(\sigma) \in G$. The resulting dual maps $\delta : C^n(X; G) \to C^{n+1}(X; G)$ are the coboundary maps. $\delta(\varphi)$ is the composition $\varphi \circ \partial : C_{n+1}(X) \to G$, so for a singular $(n+1)$-simplex $\sigma : \Delta^{n+1} \to X$ we have

$$\delta(\varphi)(\sigma) = \sum_i (-1)^i \varphi(\sigma|[v_0, \cdots, \hat{v}_i, \cdots, v_{n+1}]).$$

The $\mathbf{n^{th}}$ **singular cohomology group with coefficients in G** $H^n(X; G)$ is of course defined by $H^n(X; G) = \text{Ker }\delta_n / \text{Im }\delta_{n-1}$, where elements of $\text{Ker }\delta_n$ are **cocycles** and elements of $\text{Im }\delta_{n-1}$ are **coboundaries**. In summary, the cochain complex looks like the following:

$$\cdots \leftarrow C^{n+1}(X; G) \xleftarrow{\delta_n} C^n(X; G) \xleftarrow{\delta_{n-1}} C^{n-1}(X; G) \leftarrow \cdots \leftarrow C^0(X; G) \leftarrow 0.$$

The **Universal Coefficient Theorem for Cohomology** gives the following isomorphisms:

Theorem 2.4.2 ([Hat02], p.190). $H^n(X;\mathbb{Z}) \approx \mathrm{Hom}(H_n(X))$

In the case that $G = \mathbb{Z}$, this implies that the cohomology groups are the dual of the homology groups. In this paper, we denote the dual pairing of $\alpha \in H_n(X, \mathbb{Z})$ and $\iota H^n(X, \mathbb{Z})$ by $\langle \alpha, \iota \rangle$. We may also choose to omit notation for coefficients of the cohomology group for sake of succinctness. In these instances, we assume coefficients $G = \mathbb{Z}$.

2.4.2 Relative Cohomology

Recall that from the previous section, we have defined relative homology groups $H_n(X, A)$ for a subspace $A \subseteq X$. Analogously, we can also define its relative cohomology again by dualizing the chain complex

$$\cdots \to C_{n+1}(X, A) \xrightarrow{\partial^*_{n+1}} C_n(X, A) \xrightarrow{\partial^*_n} C_{n-1}(X, A) \to \cdots$$

where ∂^* is the induced quotient boundary map by ∂. The resulting chain complex is then

$$\cdots \leftarrow C^{n+1}(X, A; G) \xleftarrow{\delta^*_n} C^n(X, A; G) \xleftarrow{\delta^*_{n-1}} C^{n-1}(X, A; G) \leftarrow \cdots,$$

where $C^n(X, A; G) = \mathrm{Hom}(C_n(X, A), G)$. We can then define the relative cohomology groups $H^n(X, A; G) = \mathrm{Ker}\ \delta^*_n / \mathrm{Im}\ \delta^*_{n-1}$ as usual. Elements in Ker δ^*_n are called **relative cocycles** and elements in Im δ^*_{n-1} are called **relative coboundaries**.

Dualizing the short exact sequence

$$0 \to C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X, A) \to 0$$

we have for relative homology, we obtain

$$0 \leftarrow C^n(A; G) \xleftarrow{i^*} C^n(X; G) \xleftarrow{j^*} C^n(X, A; G) \leftarrow 0.$$

A simple check confirms that this is indeed another exact sequence.

2.4.3 Cellular Cohomology

For a CW complex X we can dualize Figure 2.3.1 to get:

The cellular cochain complex is then the horizontal sequence in the diagram above with coefficients in a given group G. The cellular cochain complex is isomorphic to the dual of the cellular chain complex defined previously. To see that the cellular cohomology agrees with the singular cohomology, consider the following theorem:

Theorem 2.4.3 ([Hat02], p.203). $H^n(X; G) \cong {}^{\ker d^*_n}\!/\!_{\mathrm{im}\ d^*_{n-1}}$

$$
\begin{tikzcd}[column sep=small]
 & & & 0 & & & \\
 & & H^{n-1}\left(X^{n-1}\right) \arrow[ur] \arrow[dr, "\delta_{n-1}"] & & & & \\
\cdots \arrow[r] & H^{n-1}\left(X^{n-1}, X^{n-2}\right) \arrow[ur, "j_{n-1}^{*}"] \arrow[rr, "d_{n-1}^{*}"] & & H^{n}\left(X^{n}, X^{n-1}\right) \arrow[rr, "d_{n}^{*}"] \arrow[dr, "j_{n}^{*}"'] & & H^{n+1}\left(X^{n+1}, X^{n}\right) \arrow[r] & \cdots \\
 & & & & H^{n}\left(X^{n}\right) \arrow[ur, "\delta_{n}"'] \arrow[dr] & & \\
 & & & H^{n}(X) \cong H^{n}\left(X^{n+1}\right) \arrow[ur] & & 0 & \\
 & & 0 \arrow[ur] & & & &
\end{tikzcd}
$$

Figure 2.4.1: Cellular cohomology

2.4.4 Mayer-Vietoris Sequence

For a topological space X and subspaces A, B such that X is the union of the interior of A and B, we have the following long exact sequence called the **Mayer-Vietoris Sequence**:

$$
\begin{aligned}
\cdots \rightarrow H_{n+1}(X) \xrightarrow{\partial} H_n(A \cap B) \xrightarrow{i_*, j_*} H_n(A) \oplus H_n(B) \xrightarrow{k_* - l_*} H_n(X) \xrightarrow{\partial} \cdots \\
\rightarrow H_0(X) \rightarrow 0.
\end{aligned}
$$

In the above sequence, the star-subscripted maps are maps induced by inclusion maps $i : A \cap B \rightarrow A$, $j : A \cap B \rightarrow B$, $k : A \rightarrow X$, $l : B \rightarrow X$. Notice that a class $[\alpha] \in H_{n+1}(X)$ is represented by a cycle z, and we can choose z to be the sum $x + y$ for chains x, y in A and B. Since $\partial(x + y) = 0$, we have $\partial x = -\partial y$. Therefore we can define the map $\partial : H_{n+1}(X) \rightarrow H_n(A \cap B)$ explicitly by $\partial \alpha = \partial x = -\partial y \in H_n(A \cap B)$.

There is also a relative form of the Mayer-Vietoris Sequence, which is the long exact sequence

$$
\begin{aligned}
\cdots \rightarrow H_{n+1}(X, Y) \xrightarrow{\partial} H_n(A \cap B, C \cap D) \xrightarrow{i_*, j_*} H_n(A, C) \oplus H_n(B, D) \\
\xrightarrow{k_* - l_*} H_n(X, Y) \rightarrow \cdots,
\end{aligned}
$$

where $Y \subset X$ and is the union of $C \subset A$ and $D \subset B$.

Taking the duals of the above sequences, we have the Mayer-Vietoris Sequences for cohomology:

(Absolute version)

$$\cdots \to H^n(X;G) \xrightarrow{(k_*-l_*)*} H^n(A;G) \oplus H^n(B;G) \xrightarrow{i_*^*,j_*^*} H^n(A\cap B;G) \xrightarrow{\delta} H^{n+1}(X;G) \to \cdots$$

(Relative version)

$$\cdots \to H^n(X,Y;G) \xrightarrow{(k_*-l_*)*} H^n(A,C;G) \oplus H^n(B,D;G) \xrightarrow{i_*^*,j_*^*} H^n(A\cap B, C\cap D;G) \xrightarrow{\delta} H^{n+1}(X,Y;G) \to \cdots$$

The Mayer-Vietoris sequences are useful for computations as well as proving certain properties of the topological spaces by the exactness of the sequences.

Example 2.4.4. For $k > 0$,the $k-$sphere S^k can be constructed as the interior of the union of two $k-$discs along their boundaries S^{k-1}. Since $k-$discs are contractible, their cohomology (and homology) groups are trivial. By the Mayer-Vietoris Sequence, we have:

$$\cdots \to 0 \to H^{n-1}(S^{k-1};\mathbb{Z}) \xrightarrow{\delta} H^n(S^k;\mathbb{Z}) \to 0 \to \cdots .$$

By the exactness of the sequence, δ is an isomorphism. Therefore we can compute cohomology groups of spheres by induction on n with $H^1(S^1)$ as the base case and conclude that $H^n(S^k) \approx \mathbb{Z}$ if $n = k$ and is trivial otherwise for $n > 0$. From dualizing the cellular homology group we can also see that $H^0(S^k) \approx \mathbb{Z}$ for all $k > 0$. Therefore $H^n(S^k) \approx \mathbb{Z}$ for $n = 0, k$ and is trivial otherwise.

2.4.5 Induced Homomorphisms

Previously we showed that a map $f : X \to Y$ induces chain maps $f_\# : C_n(X) \to C_n(Y)$. Now consider the dual of the chain groups $\mathrm{Hom}(C_n(X);G) = C^n(X;G)$ and $\mathrm{Hom}(C_n(Y);G) = C^n(Y;G)$. $f_\#$ then induces the cochain maps $f^\# : C^n(Y;G) \to C^n(X;G)$ which sends $\varphi \in C^n(Y;G)$ to $\varphi \circ f_\# \in C^n(X;G)$. Dualizing the relation $f_\#\partial = \partial f_\#$, we obtain the relation $\delta f^\# = f^\#\delta$, hence $f^\#$ also induces homomorphisms $f^* : H^n(Y;G) \to H^n(X;G)$. To summarize, we have the following commutative diagram:

$$\begin{tikzcd}
\cdots & C^{n+1}(X;G) \arrow[l, "\delta_{n+1}"'] & C^n(X;G) \arrow[l, "\delta_n"'] & C^{n-1}(X;G) \arrow[l, "\delta_{n-1}"'] & \cdots \arrow[l] \\
\cdots & C^{n+1}(Y;G) \arrow[l, "\delta_{n+1}"'] \arrow[u, "f^\#_{n+1}"'] & C^n(Y;G) \arrow[l, "\delta_n"'] \arrow[u, "f^\#_n"'] & C^{n-1}(Y;G) \arrow[l, "\delta_{n-1}"'] \arrow[u, "f^\#_{n-1}"'] & \cdots \arrow[l]
\end{tikzcd}$$

That is to say, a map between spaces $f : X \to Y$ induces homomorphisms between the cohomology groups of X and Y over the same coefficients.

2.4.6 Cup Product

In this section we introduce the notion of **cup product**, which is a product between cochains with coefficients in a ring R. It will be useful for constructing so-called "characteristic classes" in later sections.

Definition 2.4.5. For $\varphi \in C^k(X;R)$ and $\psi \in C^l(X;R)$, the cup product $\varphi \smile \psi \in C^{k+l}(X;R)$ assigns to a singular simplex $\sigma : \Delta^{k+l} \to X$ the value

$$(\varphi \smile \psi)(\sigma) = \varphi(\sigma|[v_0, \cdots, v_k])\psi(\sigma|[v_k, \cdots, v_{k+l}]).$$

For the cup product to be of use to us, we need it to be well-defined on cohomology classes, so it can be seen as a product $H^k(X;R) \times H^l(X;R) \to H^{k+l}(X;R)$. To do so, consider the following formula relating it to the coboundary map:

$$\delta(\varphi \smile \psi) = \delta\varphi \smile \psi + (-1)^k \varphi \smile \delta\psi$$

for $\varphi \in C^k(X;R)$ and $\psi \in C^l(X;R)$.

This formula implies that the cup product of two cocycles is again a cocycle since both summands on the right vanishes, and the cup product of a cocycle and a coboundary is a coboundary since one of the summands on the right vanishes. That is to say, the cup product is well-defined on cohomology classes.

It is worth noticing that the definition of cup product also gives relative cup products

$$H^k(X;R) \times H^l(X,A;R) \to H^{k+l}(X,A;R)$$

$$H^k(X,A;R) \times H^l(X;R) \to H^{k+l}(X,A;R)$$
$$H^k(X,A;R) \times H^l(X,A;R) \to H^{k+l}(X,A;R).$$

Regarding commutativity, we have the following:

Theorem 2.4.6 ([Hat02], p.210)**.** If R is commutative,

$$\alpha \smile \beta = (-1)^{k+l}\beta \smile \alpha$$

for all $\alpha \in H^k(X,A;R)$ and $\beta \in H^l(X,A;R)$.

We also have the useful fact that induced homomorphisms act linearly via the cup product:

Proposition 2.4.7 ([Hat02], p.210)**.** Let $f : X \to Y$ be a continuous function. Then f induces homomorphisms $f^* : H^n(Y,G) \to H^n(X,G)$. These maps satisfy $f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$ for $\alpha, \beta \in H^*(Y,G)$.

2.4.7 Gysin Sequence

The cup product is a useful tool in constructing an exact sequence known as the **Gysin Sequence**, which defined on fibre bundles $S^{n-1} \to E \xrightarrow{p} B$ satisfying some orientability hypothesis that holds true for any simply-connected base space B. For such bundles, we have the following:

$$\cdots \;\to\; H^{i-n}(B;R) \;\xrightarrow{\smile e}\; H^i(B;R) \;\xrightarrow{p^*}\; H^i(E;R) \;\to\; H^{i-n+1}(B;R) \;\to\; \cdots,$$

where R is a commutative ring, p^* is the induced map from the cohomology group of B to E by the projection map of the bundle, and $e \in H^n(B;R)$ is the "Euler class" of E which is introduced in the next section.

Notice that since $H^i(B;R) = 0$ for $i < 0$, if $i < n-1$, the exactness of the Gysin sequence gives isomorphisms

$$p^* : H^i(B;R) \approx H^i(E;R).$$

This is an important property of the Gysin sequence that will be used in the justification of exotic spheres.

2.4.8 Orientation and Poincaré Duality

In this section we will give the statement of the **Poincaré Duality Theorem**. To state the theorem we need an algebraic topological definition of orientation.

Definition 2.4.8. An n-dimensional manifold M is **orientable** if $H_n(M)$ is isomorphic to the integers $\mathbb{Z}$.

Definition 2.4.9. An **orientation** of an n-dimensional manifold M is a choice of generator of $H_n(M)$.

Now we can state the Poincaré Duality Theorem.

Theorem 2.4.10 (Poincaré Duality, [Hat02], p.239)**.** Let M be a closed, oriented n-dimensional manifold. Then there exist canonically defined isomorphisms $H^k(M;\mathbb{Z}) \to H_{n-k}(M)$.

There is an explicit expression of the isomorphism defined in terms of the **cap product**, which is beyond the scope of this paper. Nevertheless, we will make use of the result to define the "Euler class" in the next section.

2.5 Characteristic Classes

One of the most important concepts used in proving the existence of exotic spheres, in particular that these objects are not diffeomorphic to the 7-sphere, is characteristic classes.

Roughly speaking, a **characteristic class** is a machine κ that assigns to an n-dimensional vector bundle $E \to X$ a class in cohomology ring of the base space, $\kappa(E) \in H^k(X, G)$, for some fixed integers n and k and coefficients G.

In particular, characteristic classes satisfy a naturality property: $x(f^*(E)) = f^*(x(E))$ for a pullback f^*E. That is to say, if $f : Y \to X$ is a map between spaces, and $E \to X$ is a (real, oriented) vector bundle with pullback bundle $f^*E \to Y$, then $\kappa(f^*(E)) = f^*(\kappa(E)) \in H^*(Y, G)$, where we understand the left-hand side to be the characteristic class of the pullback bundle, and the right-hand side to be the image of $\kappa(E)$ under the induced (contravariant) homomorphisms $f^* : H^*(X, G) \to H^*(Y, G)$.

From this point onward, we will omit the specification of coefficients in homology/cohomology classes, assuming unless otherwise stated that we are working with integer coefficients.

2.5.1 Euler Classes

The **Euler class** is a type of characteristic class derived from a real, n-dimensional, oriented vector bundle over a closed, oriented manifold. Denoted by $e(E)$, the Euler class assigns to each real oriented vector bundle E an element of the nth cohomology group of the base manifold, with coefficients over the integers. The standard definition of the Euler class, as seen in [Hat03] p.88 or [MS75] p.98, constructs them by requiring an understanding of Thom classes and the Thom isomorphism. Since these concepts will not be used in the rest of this manuscript, we are in favour of the commonly used alternative definition detailed in Bott and Tu's *Differential Forms in Algebraic Topology* [BT13] p.72. This version, though also related to Thom classes, can be understood in a more direct manner. This is given below.

Definition 2.5.1. Suppose $\mathbb{R}^n \to E \xrightarrow{\pi} X^m$ is a real oriented rank-n vector bundle over a closed, oriented, m-dimensional manifold via projection map π. We can assign to E an element of $H^n(X, \mathbb{Z})$ via the following construction. Consider the zero section of such a vector bundle, $s_0 : X \to E$. This section is in fact an embedding of M into E. Consider another section $s : X \to E$ that intersects the zero section transversally - that is to say, such that at every point of intersection, their tangent spaces generate the entire tangent space of the manifold at that point: $T_p(s_0) + T_p(s) = T_p(X)$ [BT13] p.68. This is visualised in Figure 2.5.1

Their intersection is then an $m-n$-dimensional closed submanifold of E,

$$N = s(X) \cap s_0(X) \hookrightarrow E.$$

Let $i : N \hookrightarrow E$ denote the inclusion map. Consider the homology sequence

$$[N] \in H_{m-n}(N) \xrightarrow{i_*} H_{m-n}(E) \xrightarrow{\pi_*} H_{m-n}(X).$$

We then define the Euler class $e(E) \in H^m(X)$ to be the Poincaré dual of $\pi_* i_*([N])$.

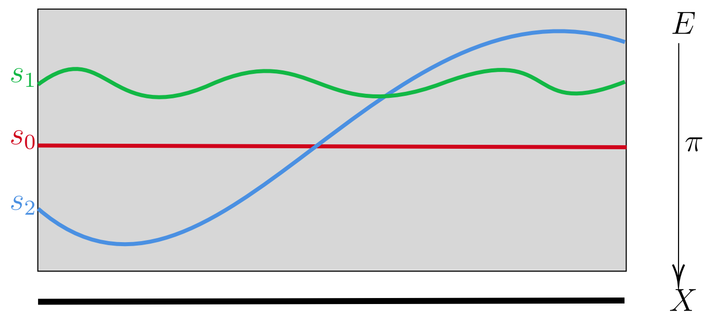


Figure 2.5.1: Sections s_1 and s_2 intersect s_0 transversally.

Note that the Euler class is independent of choice of section [BT13]. From the definition, it is clear that if a vector bundle $E \to X$ possesses a nowhere-vanishing section, then $e(E) = 0$. From this we have immediate examples of Euler classes at our disposal:

Example 2.5.2. Let $X = \mathbb{R}^n$, and take the line bundle over X. Then the total space of this bundle $E \cong \mathbb{R}^{n+1}$, with the zero section given by $s_0(x_1, ..., x_n) = (0, ..., 0)$. We can clearly choose any section that simply does not intersect with the zero section to show that $e(E) = 0$. For example, $s(x_1, ..., x_n) = (1, ..., 1)$.

This is true for any closed, oriented manifold with a trivial line bundle over it:

Example 2.5.3. Let X be a closed, oriented manifold and consider the trivial line bundle $\pi : E \to X^m$. Then $e(E) = 0$.

Even more generally, this result is also true for arbitrary characteristic classes. However, this result is only relevant to us in the cases of Euler and Pontryagin classes, the latter of which we will soon define.

Proposition 2.5.4 ([Hat03], p.91)**.** The Euler class possesses some important properties listed below.

- Naturality: $e(f^*E) = f^*e(E)$ for a pullback f^*E.
- Whitney sum: $e(E_1 \oplus E_2) = e(E_1) \smile e(E_2)$.
- Orientation: $e(E) = -e(\overline{E})$.

2.5.2 Chern Classes

Definition 2.5.5 ([Hat03], p.78)**.** Let $\mathbb{C}^n \to E \to X$ be a rank-n complex vector bundle. Then there exists a unique sequence $c_i(E)$ of cohomology classes called the Chern classes

$$c_i(E) \in H^{2i}(X, \mathbb{Z})$$

for $i = 0, 1, 2, ...$ satisfying the following properties:

- Naturality: $c_i(f^*(E)) = f^*(c_i(E))$ for a pullback f^*E.
- Denoting $c := 1 + c_1 + c_2 + ... \in H^*(X, \mathbb{Z})$, $c(E_1 \oplus E_2) = c(E_1) \smile c(E_2)$.
- $c_0(E) = 1 \in H^0(X)$ and $c_i(E) = 0$ for $i > n$.
- If E is the canonical line bundle $E \to \mathbb{C}P^1$, then $c_1(E)$ is a generator of $H^2(\mathbb{C}P^1, \mathbb{Z}) = \mathbb{Z}$.

The construction $c := 1 + c_1 + c_2 + ...$ is known as the **total Chern class**. An analogous definition exists for Pontryagin classes, as will be seen.

Because we are considering complex vector bundles, we also have an additional property of conjugation.

Definition 2.5.6. From any complex vector space V we can construct a new vector space V^{cong} by introducing conjugate scalar multiplication: for $v \in V^{\text{cong}}$ and $\lambda \in \mathbb{C}$, we define multiplication to be $\lambda \cdot v := \overline{\lambda} v$.

Performing this conjugation fibrewise, we can construct from a complex vector bundle $\mathbb{C}^n \to E \to X$ a new vector bundle $\mathbb{C}^n \to E^{\text{cong}} \to X$. We then have the following property:

Proposition 2.5.7 ([MS75], p.168)**.** Let $\mathbb{C}^n \to E \to X$ be a complex vector bundle, and $\mathbb{C}^n \to E^{\text{cong}} \to X$ its conjugate. Then $c_i(E^{\text{cong}}) = (-1)^i c_i(E)$.

In particular then, it follows that for odd i and vector bundles isomorphic to their conjugates,

$$c_i(E) = -c_i(E^{\text{cong}}) = -c_i(E) \implies c_i(E) = 0.$$

An important fact is that we can in some form consider the Chern classes of a real vector bundle by taking its **complexification**.

Definition 2.5.8. Let $\mathbb{R}^n \to E \to X$ be a real rank-n vector bundle over X. We define its complexification by $E \otimes \mathbb{C} = E \oplus iE$. We then have a complex vector bundle $\mathbb{C}^n \to E \otimes \mathbb{C} \to X$ for which we can construct Chern classes.

We can also construct a real vector bundle from a complex one by "forgetting the complex structure". That is to say, the underlying real vector bundle of a complex vector bundle $\mathbb{C}^n \to E \to M$ is a rank-$2n$ real vector bundle $\mathbb{R}^{2n} \to E_{\mathbb{R}} \to M$ where we think of each fibre as a real $2n$-dimensional vector space, rather than a complex n-dimensional vector space.

Proposition 2.5.9 ([MS75], p.173)**.** For any complex vector bundle E, the complexification of the underlying real vector bundle, $E_{\mathbb{R}} \otimes \mathbb{C}$, is canonically isomorphic to the whitney sum $E \oplus E^{\text{cong}}$.

The Chern classes of a vector bundle are analagous to the Euler class. In fact, given some $\mathbb{C}^n \to E \to X$, if $\mathbb{R}^{2n} \to E_{\mathbb{R}} \to X$ is the underlying real vector bundle, then the top Chern class is the same as the Euler class:

$$c_n(E) = e(E_{\mathbb{R}}) \in H^{2n}(X, \mathbb{Z}).$$

We also have an interesting remark that will be of use to us later:

Remark 2.5.10. Consider the trivial complex bundle over some X, $\mathbb{C}^k \times X \to X$, and denote it by ε^n. Let $E \to X$ be some other complex rank-n vector bundle. We say that E is **stably trivial** if taking the direct sum of E and ε^k for some k gives a trivial bundle: $E \oplus \varepsilon^k = \varepsilon^{n+k}$. By the properties of the Chern classes, it follows that every stably trivial bundle has trivial Chern classes [Hat03] p.10, [KP18] p. 6-7.

2.5.3 Pontryagin Classes

The final and most important characteristic classes that we will consider are the Pontryagin classes. These are constructed directly from Chern classes.

Definition 2.5.11. Let $\mathbb{R}^n \to E \to X$ be a real rank-n vector bundle over X. We define the ith Pontryagin class to be

$$p_i(E) = (-1)^i c_{2i}(E \otimes \mathbb{C}) \in H^{4i}(X, \mathbb{Z}).$$

Notice that this definition does not involve a choice of orientation. In fact, two orientations on a vector bundle give rise to the same Pontryagin classes. Relevant properties of the Pontryagin classes are listed below.

Proposition 2.5.12 ([MS75], p.175)**.** Let $\mathbb{R}^n \to X \to E$ be a real, oriented vector bundle. Then the following properties hold for its Pontryagin classes.

- Naturality: $p_i(f^*E) = f^*(p_i(E))$.
- Whitney sum: For the total Pontryagin class $p := 1 + p_1 + p_2 + \dots$, we have that $p(E_1 \oplus E_2) = p(E_1)p(E_2)$ (modulo 2-torsion).
- Orientation: $p_i(E) = p_i(\overline{E})$.

Pontryagin classes are related to Euler classes directly through the following proposition.

Proposition 2.5.13 ([Hat03], p.94)**.** Let $\mathbb{R}^{2n} \to E \to X$ be an oriented, real rank-$2n$ vector bundle with Euler class $e(E) \in H^{2n}(X, \mathbb{Z})$. Then $p_n(E) = e(E)^2$.

We also have a useful identity relating all of the Pontryagin classes of a bundle to its total Chern class.

Proposition 2.5.14 ([MS75], p.177)**.** The Pontryagin classes of a bundle is determined completely by its Chern classes in the following way

$$1 - p_1 + p_2 - ... \pm p_n = (1 - c_1 + c_2 - ... \pm c_n)(1 + c_1 + c_2 + ... + c_n).$$

Finally, we have an analagous result to Remark 2.5.10 for Pontryagin classes:

Remark 2.5.15. By definition of Pontryagin classes and Remark 2.5.10, the Pontryagin classes of a stably trivial vector bundle vanish.

A key case of the above that we will use in the next section is the following:

Example 2.5.16 ([Hat03] p.81, [KP18] p.6-7)**.** The tangent bundle to S^n is stably trivial, as taking the direct sum of TS^n with a normal bundle to S^n in $\mathbb{R}^{n+1}$, which is itself a trivial line bundle, gives us a trivial line bundle. That is, $TS^n \oplus \varepsilon^1 = \varepsilon^{n+1}$.

Chapter 3

Exotic Spheres

Having established the necessary background material needed to understand Milnor's exotic spheres, we may now turn our attention to proving their existence. That is, we will prove the following theorem

Theorem 3.0.1 (Exotic Spheres)**.** There exist 7-manifolds that are homeomorphic but not diffeomorphic to the 7-sphere, S^7.

We will approach the proof of this result in the following way. First, we will construct candidates for exotic spheres. We will construct them as sphere bundles over S^4 via what is known as the clutching construction. We will then prove that some of these candidates are homeomorphic to S^7 by an application of Morse Theory. We will then conclude the proof by showing that some of these candidates are not diffeomorphic to S^7 by considering non-solutions to a specific identity.

3.1 Constructing candidates for Exotic Spheres

In this section we will discuss a method for constructing candidates for exotic spheres. We will construct them as the total space of some sphere bundles over S^4. We first need to introduce some concepts that will allow us to construct these bundles. The beginning of this section is adapted from [Hat03].

The first tool that we will need is the clutching construction. This is a way of constructing vector bundles over spheres. The clutching construction procedure is as follows: we can decompose S^n as the union of its upper and lower hemispheres, which are homeomorphic to disks: $S^n = D^n_+ \cup D^n_-$ where the boundary is $D^n_+ \cap D^n_- = S^{n-1}$. Then for some map $\varphi : S^{n-1} \to \mathrm{GL}_k(\mathbb{R})$, we define the quotient space

$$E_\varphi = (D^n_- \times \mathbb{R}^k) \sqcup (D^n_+ \times \mathbb{R}^k)/\sim$$

where we identify $(x, v) \in \partial D^n_- \times \mathbb{R}^k$ with $(x, \varphi(x)v) \in \partial D^n_+ \times \mathbb{R}^k$. Note that we apply the linear map $\varphi(x) \in \mathrm{GL}_k(\mathbb{R})$ to $v \in \mathbb{R}^k$ to get another element of $\mathbb{R}^k$. We can see from the way the identification is defined that this "gluing" of the trivial bundles is the identity on the boundary of the hemispheres, but the map φ gives a 'twist' in the fibres $\mathbb{R}^k$. This is illustrated in Figure 3.1.1.

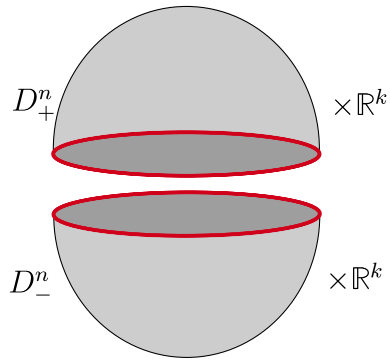


Figure 3.1.1: Constructing E_φ by identifying bundles of discs along boundaries.

This gives us a k-dimensional vector bundle

$$\begin{tikzcd} \mathbb{R}^k \arrow[r, hook] & E_\varphi \arrow[d, "\pi"] \\ & S^n = D^n_- \cup D^n_+ \end{tikzcd}$$

where the map π is projection onto the first coordinate, $\pi(x, v) = x$. We call the map φ a **clutching function** corresponding to the vector bundle above.

Definition 3.1.1. Denote $GL^+_k(\mathbb{R}) := \{A \in \mathrm{Mat}_k(\mathbb{R}) : \det(A) > 0\}$.

Definition 3.1.2. Denote by $\mathrm{Vect}^n_+(X)$ the set of isomorphism classes of oriented n-dimensional vector bundles over the space X.

Proposition 3.1.3. Let E_φ and E_ψ be two vector bundles constructed as above. Then they are isomorphic if φ and ψ are homotopic.

Proof. We give a proof adapted from a discussion given in [Hat03], p.23. Suppose the maps $\varphi : S^{n-1} \to GL_k(\mathbb{R})$ and $\psi : S^{n-1} \to GL_k(\mathbb{R})$ are homotopic via the homotopy $F : S^{n-1} \times I \to GL_k(\mathbb{R})$. We can use this homotopy as a clutching function to construct a vector bundle. Define the quotient

$$E_F = (D^n_- \times I \times \mathbb{R}^k) \sqcup (D^n_+ \times I \times \mathbb{R}^k)/\sim$$

where the equivalence relation now identifies $(x, t, v) \in \partial D^n_- \times I \times \mathbb{R}^k$ with $(x, t, F(x, t)v) \in \partial D^n_- \times I \times \mathbb{R}^k$. The vector bundle

$$\begin{tikzcd} E_F \arrow[d] \\ S^n \times I \end{tikzcd}$$

then restricts to E_φ over $S^n \times \{0\}$ and to E_ψ over $S^n \times \{1\}$. By Proposition 1.7 in [Hat03] these vectors bundles are then isomorphic. □

This proposition allows us to construct a well-defined bijection from the set of homotopy classes of maps from $S^{n-1} \to GL_k^+(\mathbb{R})$, denoted $[S^{n-1}, GL_k^+(\mathbb{R})]$ to $\mathrm{Vect}_+^k(S^n)$, [Hat03], p.25. This map sends the homotopy class of the clutching map $[\varphi]$ to E_φ.

Denote this bijection by $\Phi : [S^{n-1}, GL_k^+(\mathbb{R})] \to \mathrm{Vect}_+^k(S^n)$. Since we want to construct sphere bundles over S^4, we need to consider the special case $k = n = 4$. We can simplify the above proposition further by consdering homotopy classes of maps $S^{n-1} \to SO(k)$, where $SO(k)$ is the subgroup of $GL_k^+(\mathbb{R})$ in which the matrices have determinant 1. This group is known as special orthogonal group or the rotation group.

The above simplification gives us a bijection $\Psi : [S^3, SO(4)] \to \mathrm{Vect}_+^4(S^4)$ where we have taken $n = k = 4$. Combining this with Remark 2.4.5, we have that $\pi_3(SO(4)) = [S^3, SO(4)] \cong \mathrm{Vect}_+^4(S^4)$. This tells us that in order to construct the bundles we want, we must calculate $\pi_3(SO(4))$. To do this we will need the following result:

Proposition 3.1.4. [[Hat02], p.342] A covering space projection $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ induces isomorphisms $p_* : \pi_n(\tilde{X}, \tilde{x}_0) \to \pi_n(X, x_0)$ for all $n \geq 2$.

Proposition 3.1.5. $\pi_3(SO(4)) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Proof. By the previous proposition, it suffices to construct a covering space projection $p : S^3 \times S^3 \to SO(4)$, as then we will have $\pi_3(S^3 \times S^3) \cong \pi_3(SO(4))$. By Proposition 2.2.9, $\pi_3(S^3) \cong \mathbb{Z}$ so $\pi_3(S^3 \times S^3) \cong \mathbb{Z} \oplus \mathbb{Z} \cong \pi_3(SO(4))$.

The following is based on [McE15]. Define a homomorphism $\eta : S^3 \times S^3 \to SO(4)$ which takes a pair of unit quaternions (p, q), which we identify with S^3, to the linear map $\psi_{p,q} : \mathbb{H} \to \mathbb{H}$ defined by $\psi_{p,q}(x) = pxq^{-1}$. Consider the kernel of this map, $\ker\eta = \{(p, q) \in S^3 \times S^3 \cong \mathbb{H} \times \mathbb{H} : px = xq \text{ for all } x \in \mathbb{H}\}$. Setting $x = 1$, we get $(p, q) \in \ker\eta$ if $p = q$. This implies that p must be in the centre of $\mathbb{H}$, which is equal to $\mathbb{R} \in \mathbb{H}$ [Con]. But since we are considering unit quaternions, this means that the centre is equal to $\{\pm 1\}$. Therefore $\ker\eta = \{(1, 1), (-1, -1)\}$. So $S^3 \times S^3$ is a double cover of $SO(4)$. We can conclude that η is a covering map and apply the previous proposition to get that $\pi_3(SO(4)) \cong \mathbb{Z} \oplus \mathbb{Z}$. □

Note that $SO(n)$ also acts on D^n and S^{n-1}. So, if instead of taking the clutching functions over $\mathbb{R}$ we restrict to some discs or spheres, then we get a disc or sphere bundle associated to the vector bundle that arises via the clutching construction.

Therefore to construct candidates for exotic spheres, we will consider S^3 bundles over S^4 with structure group $SO(4)$. We can classify these vector bundles using $\pi_3(SO(4)) \cong \mathbb{Z} \oplus \mathbb{Z}$.

To construct these bundles, define a clutching function by

$$f_{i,j} : S^3 \to SO(4)$$

where $f_{i,j}$ takes a unit quaternion q (where we identify the unit quaternions with S^3 as in section 2.1.1) and sends it to the linear map $v \mapsto q^i v q^j$ ([Mil56], p.402) for some $v \in \mathbb{R}^4 \cong \mathbb{H}$, where we use quaternionic multiplication on the right hand side. Then for each $(i,j) \in \mathbb{Z} \oplus \mathbb{Z}$, we get a sphere bundle

$$\begin{tikzcd} S^3 \arrow[r, hook] & E_{i,j} \arrow[d] \\ & S^4 \end{tikzcd}$$

where the candidates for the exotic spheres are the total spaces of these bundles, E_{ij}.

3.2 E_{ij} is homeomorphic to S^7

To prove that some of the candidates E_{ij} are homeomorphic to S^7, we need a theorem from a branch of differential topology called **Morse Theory**.

Theorem 3.2.1 (Reeb's Sphere Theorem)**.** If M is a compact manifold and f is a real valued differentiable function on M with only two critical points, both of which are non-degenerate, then M is homeomorphic to a sphere.

We now give a proof of this theorem, which is due to [Mil73].

Lemma 3.2.2 ([Mil73], p.10)**.** A smooth vector field X on a manifold M which vanishes outside a compact set $K \subset M$ generates a unique one-parameter group of diffeomorphisms of M, φ_t, which satisfies

$$\frac{d\varphi_t(q)}{dt} = X_{\varphi_t(q)}, \quad \varphi_0(q) = q$$

for all $q \in M$.

This lemma leads to a theorem that is crucial for the proof of Theorem 3.2.1.

Theorem 3.2.3. Let f be a smooth real valued function on a manifold M. Let $a < b$ and suppose that the set $f^{-1}[a,b]$ is compact, and contains no critical point of f. Then $M^a := f^{-1}(-\infty, a]$ is diffeomorphic to $M^b := f^{-1}(-\infty, b]$.

Proof. Choose a Riemannian metric on M, and denote it as $\langle X, Y \rangle$. The gradient of f is defined to be the vector field $\mathrm{grad} f$ on M which is characterised by the identity

$$\langle X, \mathrm{grad} f \rangle = X(f)$$

for any vector field X. Let $c : \mathbb{R} \to M$ be a curve with velocity $\frac{dc}{dt}$. Then

$$\left\langle \frac{dc}{dt}, \mathrm{grad} f \right\rangle = \frac{d(f \circ c)}{dt}.$$

Next let $\rho : M \to \mathbb{R}$ be a smooth function which is equal to $1/\langle \mathrm{grad} f, \mathrm{grad} f\rangle$ throughout the compact set $f^{-1}[a,b]$ and vanishes outside of a compact neighborhood of this set. This definition is valid since grad f only vanishes at critical points of f on M. By Lemma 3.2.2, the vector field $X(q) = \rho(q)(\mathrm{grad} f)_q$ for $q \in M$ generates a one-parameter group of diffeomorphisms $\varphi_t : M \to M$.

For a fixed $q \in M$, consider the function $t \to f(\varphi_t(q))$. If $\varphi_t(q)$ lies in the set $f^{-1}[a,b]$,

$$\frac{df(\varphi_t(q))}{dt} = \left\langle \frac{d\varphi_t(q)}{dt}, \mathrm{grad} f \right\rangle = \langle X, \mathrm{grad} f\rangle = 1.$$

The second equality follows from how φ_t is generated by X in Lemma 3.2.2. The last equality follows from the definition of X and ρ. Therefore the correspondence

$$t \to f(\varphi_t(q))$$

is linear with derivative 1 as long as $f(\varphi_t(q))$ lies between a and b.

Now consider the diffeomorphism $\varphi_{b-a} : M \to M$. This carries M^a diffeomorphically onto M^b, for if $f(q) = a$,

$$f(\varphi_{b-a}(q)) = a + b - f(\varphi_0(q)) = a + b - a = b.$$

□

We also need a special case of this Morse Lemma when the critical point is a local minimum for a function on a manifold.

Lemma 3.2.4 (Morse lemma on minimum points [Mil73]p.6)**.** Let p be a non-degenerate critical point that is a local minimum of a differentiable real valued function on a manifold M. Then there is a local coordinate system $(y^1, y^2, ..., y^n)$ in a neighborhood U of p with $y^i(p) = 0$ for all i such that the identity

$$f = f(p) + (y^1)^2 + \cdots + (y^n)^2$$

holds throughout U.

Now we are ready to prove Theorem 3.2.1.

Proof. The two critical points must be the minimum and maximum of points of f. Let p be the minimum point and q be the maximum point. We can always scale f so that $f(p) = 0$ and $f(q) = 1$. By Lemma 3.2.4, there is a local coordinate system in a neighbourhood U of p such that $f = (y^1)^2 + \cdots + (y^n)^2$. Therefore for small enough ϵ_1, there is a closed neighbourhood $M^{\epsilon_1} = f^{-1}[0, \epsilon_1] \subset U$ of p that is homeomorphic to the closed unit ball in $\mathbb{R}^n$. Considering the function $g = 1 - f$, we can also find such a closed neighbourhood $f^{-1}[1 - \epsilon_2, 1]$ of q.

Since p and q are the only critical points, by Theorem 3.2.3, M^{ϵ_1} is homeomorphic to $M^{1-\epsilon_2}$. Therefore M is the union of two closed unit balls in $\mathbb{R}^n$ identified along their common boundary. Thus M is homeomorphic to the n-sphere S^n. □

Now we just need to construct a real valued differentiable function on $E_{i,j}$ which has two non-degenerate critical points. All multiplications in this construction are quaternion multiplications. Recall that we denote the real component of a quaternion q by $R(q)$.

Theorem 3.2.5. $E_{i,j}$ is homeomorphic to S^7 when $i + j = 1$.

Proof. For convenience we will construct $E_{i,j}$ in a different way that is equivalent to the clutching construction in the previous section. Equivalence is justified by Chapter 18 of [Ste51], as the identifying map below restricted to the equator S^3 of S^4 is identical to the clutching maps in the previous section for each i, j.

Notice that taking the complement of the north pole and the complement of the south pole in S^4, we can obtain two copies of $\mathbb{R}^4$ by stereographic projection. Now take two copies of $\mathbb{R}^4 \times S^3$ and identify the subsets $(\mathbb{R}^4 - \{0\}) \times S^3$ by identifying (u, v) in the first copy with

$$(u', v') = \left(\frac{u}{\|u\|^2}, \frac{u^i v u^j}{\|u\|} \right)$$

in the second copy. This identification then gives us our original candidates, E_{ij}.

Notice that under the stereographic projection, a point in $S^4 - \{\text{the poles}\}$ projected to u under one projection is projected to $u/\|u\|^2$ under the other projection. Therefore the identification of u and u' above is just "undoing" the projection. For the identification of v and v' to make sense (i.e. $\|v'\| = 1$), we need $i + j = 1$. This is the condition stated in the claim.

We can now construct our function on E_{ij}. Define the function $f : E_{ij} \to \mathbb{R}$ by

$$f(x) = \frac{R(v)}{(1 + \|u\|^2)^{\frac{1}{2}}}$$

on the first copy and

$$f(x) = \frac{R(u'')}{(1 + \|u''\|^2)^{\frac{1}{2}}}$$

on the second copy, where $u'' = u'(v')^{-1}$. For this definition to hold, we need to check that they are equivalent on the subsets where the copies are identified. Firstly, we have

$$\begin{aligned} \|u''\| &= \frac{\|u\|}{\|u\|^2} \cdot \frac{1}{\|v'\|^2} \\ &= \frac{1}{\|u\|}. \end{aligned}$$

Therefore

$$\frac{R(u'')}{(1 + \|u''\|^2)^{\frac{1}{2}}} = \frac{R(u'')\|u\|}{(1 + \|u\|^2)^{\frac{1}{2}}},$$

so we are left to prove that

$$R(u'')\|u\| = R(v)$$

on the overlap. Indeed, we have

$$\begin{aligned}
R(u'')\|u\| &= R(\frac{u}{\|u\|^2} \cdot \overline{v'})\|u\| \\
&= R(u \cdot \overline{v'})\frac{1}{\|u\|} \\
&= R\left(\frac{u}{\|u\|} \cdot \overline{\frac{u^i v u^j}{\|u\|}}\right) \\
&= R\left(\frac{u}{\|u\|} \cdot \overline{\frac{u^i}{\|u\|^i} v \frac{u^j}{\|u\|^j}}\right) \\
&= R\left(\frac{u}{\|u\|} \cdot \left(\frac{u}{\|u\|}\right)^{-j} v^{-1} \left(\frac{u}{\|u\|}\right)^{-i}\right) \\
&= R\left(\left(\frac{u}{\|u\|}\right)^{i} v^{-1} \left(\frac{u}{\|u\|}\right)^{-i}\right) \\
&= R(v^{-1}) \\
&= R(v).
\end{aligned}$$

Now we check that $f(x)$ has exactly two critical points. We express $f(x)$ by the standard coordinates $u = (u_1, u_2, u_3, u_4), v = (v_1, v_2, v_3, v_4)$ on the first copy:

$$f(x) = \frac{(1 - v_2^2 - v_3^2 - v_4^2)^{\frac{1}{2}}}{(1 + (u_1^2 + u_2^2 + u_3^2 + u_4^2)^2)^{\frac{1}{2}}}.$$

By directly computing partial derivatives, we get two critical points $(u, v) = (0, \pm 1)$. Similarly,

$$f(x) = \frac{u_1''}{(1 + ((u_1'')^2 + (u_2'')^2 + (u_3'')^2 + (u_4'')^2)^2)^{\frac{1}{2}}}$$

on the second copy, and there is no critical point on this copy. Therefore $f(x)$ has exactly two critical points.

We can check that these critical points are non-degenerate by computing the second derivatives. Such a computation can be found in [Bog11]. In particular,

$$\frac{\partial^2 f}{\partial(u_i)^2} = 1, \ \frac{\partial^2 f}{\partial(v_i)^2} = 1,$$

and

$$\frac{\partial^2 f}{\partial u_i \partial u_j} = \frac{\partial^2 f}{\partial v_i \partial v_j} = \frac{\partial^2 f}{\partial u_i \partial v_j} = 0$$

at both critical points. Therefore the Hessian matrices are the diagonal matrix $diag(1, 1, 1, 1, -1, -1, -1, -1)$, which is clearly nonsigular. Now the claim follows from Theorem 3.2.1. $\square$

It is worthwhile to briefly comment on alternative proofs that the sphere bundles E_{ij} are homeomorphic to S^7. A notably more modern and direct proof of the result is offered in [BK20], based on the generalised Poincaré conjecture. Bustamante makes use of cohomology and homotopy entirely, and does not introduce any results of Morse Theory. However, we believe that, for the sake of exposition, it would be a shame to not stay true to the Milnor's method.

3.3 A condition for when E_{ij} is diffeomorphic to S^7

In Section 3.1, we found potential candidates for exotic spheres by constructing S^3 bundles over S^4 using clutching maps. We saw that these maps were identified by $\mathbb{Z} \times \mathbb{Z}$. In Section 3.2 we saw that certain pairs (i, j) give rise to sphere bundles whose total spaces were homeomorphic to S^7. We finally complete Milnor's proof by showing that some of these topological 7-spheres cannot be diffeomorphic to the 7-sphere.

3.3.1 A note on approaches to determining diffeomorphism conditions

Proving this last statement is the crux of the proof. Oddly, it is also where Milnor starts his argument, which runs as follows. Milnor begins by constructing an invariant up to diffeomorphism of real, oriented, closed 7-manifolds. Then after constructing candidates E_{ij} for exotic spheres and determining those that are homeomorphic to S^7, Milnor further develops this invariant, constructing an identity in i and j that must be satisfied in order for them to be diffeomorphic to S^7. Inevitably this identity gives rise to topological 7-spheres that are not differentiable 7-spheres, namely E_{ij} where i and j do not satisfy it.

We will follow the work of Milnor closely in order to effectively unpack his argument. Our approach will be as follows.

1. First, we will define an important invariant: the signature of a manifold. We will explore some of its properties.
2. We will then construct Milnor's invariant $\lambda(M^7)$ for an arbitrary 7-manifold M^7. We will prove that it is in fact an invariant of M^7, even though it's definition may suggest otherwise.
3. We will calculate $\lambda(E_{ij})$ explicitly for our exotic sphere candidates. The specific way we will do this will be discussed later. This will give rise to our identity, which will allow us to conclude that there exist topological 7-spheres that are not differential 7-spheres. We will end this section with some interesting remarks.

In an attempt to provide some sort of direction for this section, below is the theorem we will prove:

Theorem 3.3.1. Let E_{ij} ($(i,j) \in \mathbb{Z} \times \mathbb{Z}$) be a real, oriented, closed manifold as constructed above, with $i + j = 1$ so that E_{ij} is a topological 7-sphere. Suppose E_{ij} is diffeomorphic to the 7-sphere. Then $(i-j)^2 \equiv \pm 1 \mod 7$.

Before we begin to prove this result, we will mention some useful sources offer alternative proofs. Of course, it is natural in the 60+ years since publication that some interesting alternatives proofs have been arisen. The first and arguably most approachable proof we have come across is given in the ICMS Lecture series on Exotic Spheres, 2020 [BK20]. In the series, Bustamante calculates the signature of the potential exotic spheres in two ways: first using the definition, and then the Hirzebruch Signature Theorem. Comparison of these calculations immediately gives the desired identity. The direction of Bustamante's argument is much more apparent than Milnor's, though the underlying mathematics is similar.

Other good explantions can be found in Braddell's *Applications of Characteristic Classes and Milnor's Exotic Spheres* [Bra16], de Melo's *Exotic 7-spheres* [dM18], and McEnroe's *Milnor's construction of exotic 7-spheres* [McE15]. Each paper was written at graduate student level, making them somewhat more accessible. We will draw from these sources in different sections to provide a comprehensive overview of Milnor's work while still remaining true to the original argument of the paper.

With this established, let us begin our work.

3.3.2 The signature of a manifold

First we need to add a specific construction to our tool kit: the signature of a manifold.

Definition 3.3.2. Let X^{4k} be a compact, oriented manifold of dimension $4k$. Define a form as follows

$$\Sigma_X : H^{2k}(X) \setminus \{\text{torsion}\} \times H^{2k}(X) \setminus \{\text{torsion}\} \to \mathbb{Z}.$$

Which maps

$$\Sigma_X : (\alpha, \beta) \mapsto \langle [\mu], \alpha \smile \beta \rangle,$$

where $[\mu]$ is a chosen fundamental class in $H_{4k}(X)$. We assume a choice of basis of $H^{2k}(X) \setminus \{\text{torsion}\}$ such that the form is diagonal over $\mathbb{Z}$. By properties of the cup product, this is actually a symmetric bilinear form. We define the signature σ of the manifold to be the number of positive eigenvalues minus the number of negative eigenvalues of this form [MS75].

The signature of a manifold has some nice properties.

Proposition 3.3.3 ([MS75], p.224)**.** Let X^{4k} be a $4k$-dimensional (real, oriented) manifold, and Y^{4l} a $4l$-dimensional (real, oriented) manifold. Then $\sigma(X+Y) = \sigma(X) + \sigma(Y)$, and $\sigma(X \times Y) = \sigma(X) \times \sigma(Y)$.

Proposition 3.3.4. The absolute value of the signature of a manifold is homotopy invariant. That is to say, the signature of a manifold is invariant up to any orientation-preserving homotopy.

Proof. This result was originally proven in René Thom's *Quelques propriétés globales des variétés différentiables* in 1954 [Tho54]. For a slightly more modern explanation of homotopy invariance, see p. 224-226 of [MS75]. □

Notice that by definition, a change in orientation induces a change in the sign of σ. This is why we must specify absolute value/orientating-preserving invariance.

Now, we can of course determine the signature of a manifold using the definition, but in more complicated or generalised cases this can get a bit tricky. Luckily we have a formula for σ which makes use of the Pontryagin class of the manifold. This formula is due to Hirzebruch [Hir54], but can also be found in [MS75] p.224.

Proposition 3.3.5 (Hirzebruch's Signature Theorem)**.** Let X^{4k} be a real, compact, oriented, $4k$-dimensional manifold. Denote a choice of fundamental class of X by $[\mu] \in H_{4k}(X)$. . Then there exists a polynomial L in $p_1, ..., p_k$ such that the signature of X^{4k} is given by

$$\sigma(X) = \langle [\mu], L(p_1, ..., p_k) \rangle,$$

where p_i denotes the ith Pontryagin class of the tangent bundle over X.

The case that will be of use to us is when $k = 2$:

Corollary 3.3.6. Let X^8 be a real, compact, oriented, 8-dimensional manifold. Then the signature of X^8 is given by

$$\sigma(X) = \frac{7}{45}\langle [\mu], p_2(X) \rangle - \frac{1}{45}\langle [\mu], p_1^2(X) \rangle.$$

3.3.3 Constructing Milnor's invariant

To construct the invariant, we will first consider arbitrary 7-manifolds M^7. We will consider the invariant in the explicit case of our E_{ij} in the next section.

Let M^7 be a closed, oriented 7-manifold. By definition of orientability, there exists a distinguished element $[\mu] \in H_7(M^7)$ that is the generator of the top homology group. By Thom [Tho54], any closed 7-manifold M^7 is the boundary of an 8-manifold N^8. We can then determine an orientation on the pair (N^8, M^7), which we will denote by $[\nu] \in H_8(N^8, M^7)$, by choosing a $[\nu]$ such that $\partial[\nu] = [\mu]$.

Applying the long exact sequence in cohomology to the pair (N^8, M^7),

$$\cdots \to H^3(M^7) \to H^4(N^8, M^7) \to H^4(N^8) \to H^4(M^7) \to \ldots$$

and assuming that $H^3(M^7) = H^4(M^7) = 0$, it follows that the map

$$i : H^4(N^8, M^7) \to H^4(N^8) \tag{3.1}$$

is an isomorphism. This allows us to define the "Pontryagin number" of a manifold N^8 with boundary M^7.

$$q(N^8) := \langle [\nu], (i^{-1}p_1(N^8))^2 \rangle,$$

where $p_1(N_8)$ denotes the first Pontryagin class of the tangent bundle of N^8.

Remark 3.3.7. Previously, we used the notation $p_1(E) \in H^4(X)$ to mean the first Pontryagin class associated with the bundle $E \to X$. We now adopt the slight abuse of notation $p_1(N_8)$ to actually mean $p_1(TN_8)$. Both forms of notation are common, and simply depend on whether we are discussing the Pontryagin classes contained in the cohomology class of a specific manifold, or if we are discussing a specific bundle. Unless stated otherwise, we will assume this new notation.

With the Pontryagin number of a manifold defined, we can immediately construct Milnor's invariant:

Definition 3.3.8. We define $\lambda(M^7) := 2q(N^8) - \sigma(N^8) \mod 7$, where $\sigma(N^8)$ is the signature of N^8.

Remark 3.3.9. In Milnor's original paper, he defines $\lambda(M^7) := 2q(N^8) - \tau(N^8)$, where $\tau(N^8)$ is the index of the quadratic form over the group $H^4(N^8, M^7)/\{\text{torsion}\}$ given by $[\alpha] \mapsto \langle [\nu], [\alpha^2] \rangle$. But this is exactly the signature of N^8, as the associated symmetric bilinear form of this quadratic form is the same as in Definition 3.3.2. We can (and will) prove results via the quadratic form and via the symmetric bilinear form interchangeably.

Remark 3.3.10. This definition of $\lambda(M^7)$ may seem rather odd, considering that we are attempting to define an invariant of M^7 by making a choice of N^8. Remarkably though, this choice of N^8 is irrelevant.

Proposition 3.3.11. $\lambda(M^7)$ is independent of choice of N^8.

Proof. Suppose that N_1^8 and N_2^8 are two 8-manifolds with common boundary M^7. To prove the claim it suffices to show that the two versions of $\lambda(M^7)$ that arise from using N_1^8 and N_2^8 are equivalent modulo 7. Let their orientations be denoted by $[\nu_1]$ and $[\nu_2]$ respectively. We can construct a new 8-manifold C^8 by smoothly gluing N_1^8 and N_2^8 along their boundary, preserving their differentiable structure.

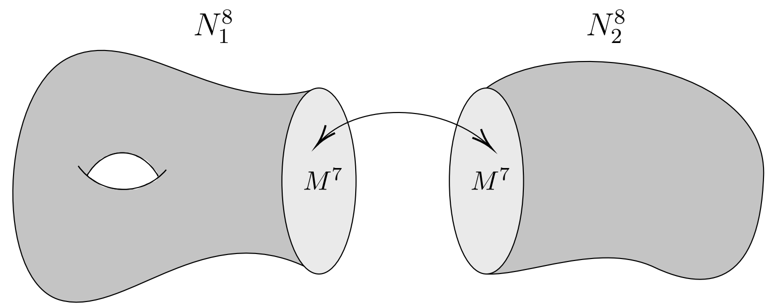


Figure 3.3.1: Gluing $C^8 := N_1^8 \cup_{M^7} N_2^8$.

Remark that in order for the gluing to occur smoothly, we must reverse the orientation of one of the 8-manifolds. We then choose an orientation for C^8 that respects the orientations $[\nu_1]$ and $[\nu_2]$, say $[\nu] \in H^8(C^8)$.

We can consider both the normal and relative Mayer-Vietoris sequnces for cohomology for C^8. For the normal Mayer-Vietoris sequence, we have

$$\cdots \to H^n(C^8) \to H^n(N_1^8) \oplus H^n(N_2^8) \to H^n(N_1^8 \cap N_2^8) \to \ldots.$$

For the relative Mayer-Vietoris sequence, we consider the pair $(C^8, M^7) = (N_1^8 \cup N_2^8, M^7 \cup M^7)$. Then we have

$$\cdots \to H^n(C^8, M^7) \to H^n(N_1^8, M^7) \oplus H^n(N_2^8, M^7) \to H^n(N_1^8 \cap N_2^8, M^7) \to \ldots.$$

By the fact that N_1^8 and N_2^8 are disjoint, it follows that

$$k : H^4(C^8) \to H^n(N_1^8) \oplus H^4(N_1^8)$$

and

$$h : H^4(C^8, M^7) \to H^4(N_1^8, M^7) \oplus H^4(N_2^8, M^7)$$

are isomorphisms.

Using equation 3.1 for the 8-manifolds N_1^8, N_2^8 and C^8, these maps arrange into a commuting diagram:

$$\begin{tikzcd}
H^4\left(C^8, M^7\right) \arrow[r, "h"] \arrow[d, "j"'] & H^4\left(N_1^8, M^7\right) \oplus H^4\left(N_2^8, M^7\right) \arrow[d, "i_1 \oplus i_2"] \\
H^4\left(C^8\right) \arrow[r, "k"'] & H^4\left(N_1^8\right) \oplus H^4\left(N_2^8\right)
\end{tikzcd}$$

If $\alpha := jh^{-1}(\alpha_1 \oplus \alpha_2) \in H^4(C^8)$ for $\alpha_1 \in H^4(N_1^8, M^7)$ and $\alpha_2 \in H^4(N_2^8, M^7)$, then it follows that

$$\langle [\nu], \alpha^2 \rangle = \langle [\nu], jh^{-1}(\alpha_1^2 \oplus \alpha_2^2) \rangle = \langle [\nu_1] \oplus -[\nu_2], \alpha_1^2 \oplus \alpha_2^2 \rangle = \langle [\nu_1], \alpha_1^2 \rangle - \langle [\nu_2], \alpha_2^2 \rangle \qquad (3.2)$$

Remark 3.3.9 then implies that

$$\sigma(C^8) = \sigma(N_1^8) - \sigma(N_2^8). \tag{3.3}$$

Now denote $\alpha_1 := i_1^{-1}p_1(N_1^8) \in H^4(N_1^8, M^7)$ and $\alpha_2 := i_2^{-1}p_1(N_2^8) \in H^4(N_2^8, M^7)$. Via the isomorphism k and naturality we have that $k(p_1(C^8)) = p_1(N_1^8) \oplus p_2(N_2^8)$. But this says that $k(p_1(C^8)) = (i_1 \oplus i_2)(\alpha_1 \oplus \alpha_2)$. By the commuting square above, we know that $i_1 \oplus i_2 = kjh^{-1}$, so $i_1 \oplus i_2(\alpha_1 \oplus \alpha_2) = kjh^{-1}(\alpha_1 \oplus \alpha_2)$. Combining these then, we must have that $p_1(C^8) = jh^{-1}(\alpha_1 \oplus \alpha_2)$. Using the exact same calculation as in Equation 3.2, it follows that

$$\langle [\nu], p_1^2(C^8)\rangle = \langle [\nu_1], \alpha_1^2\rangle - \langle [\nu_2], \alpha_2^2\rangle,$$

or equivalently

$$q(C^8) = q(N_1^8) - q(N_2^8). \tag{3.4}$$

By Hirzebruch's Signature Theorem from Corollary 3.3.6, we have that

$$\begin{aligned}
\sigma(C^8) &= \frac{7}{45}\langle [\nu], p_2(C^8)\rangle - \frac{1}{45}\langle [\nu], p_1^2(C^8)\rangle \\
\iff 45\sigma(C^8) &= 7\langle [\nu], p_2(C^8)\rangle - \langle [\nu], p_1^2(C^8)\rangle \\
\iff 45\sigma(C^8) &= 7\langle [\nu], p_2(C^8)\rangle - q(C^8) \\
\implies 45\sigma(C^8) + q(C^8) &\equiv 0 \mod 7 \\
\iff 2q(C^8) - \sigma(C^8) &\equiv 0 \mod 7.
\end{aligned}$$

Combining this last line with Equations 3.3 and 3.4, we get that

$$\begin{aligned}
&2q(C^8) - \sigma(C^8) \equiv 0 \mod 7 \\
\iff &2(q(N_1^8) - q(N_2^8)) - (\sigma(N_1^8) - \sigma(N_2^8)) \equiv 0 \mod 7 \\
\iff &2q(N_1^8) - \sigma(N_1^8) \equiv 2q(N_2^8) - \sigma(N_2^8) \mod 7
\end{aligned}$$

Which says exactly that our choice of N_1^8 or N_2^8 is irrelevant in defining $\lambda(M^7)$. □

Remark 3.3.12. The result in Equation 3.3 may seem somewhat obvious, considering the fact that the signature of a disjoint union $N_1 \cup N_2$ is the sum of the signatures of N_1 and N_2, and a reversal in orientation $(-[\nu])$ simply induces a change in sign of signature [Kre05]. The above argument is much more rigorous, however.

Corollary 3.3.13. Reversing the orientation of M^7 changes the sign of $\lambda(M^7)$.

Proof. Notice that a change in orientation of M^7 will induce a change in orientation of our manifold N^8 to $-[\nu]$. The Pontryagin number will then pick up a minus sign, and by the above the signature will also pick up a minus sign. □

Remark 3.3.14. Notice that by our construction, two diffeomorphic 7-manifolds will give rise to the same invariant. That is, if M_1^7 is diffeomorphic to M_2^7, then $\lambda(M_1^7) = \lambda(M_2^7)$.

Noting that S^7 (in fact, any S^n) has an orientation-reversing diffeomorphism onto itself, we can combine these to give the following corollary:

Corollary 3.3.15. $\lambda(S^7) \equiv 0 \mod 7$.

The construction of this invariant is relatively straightforward. The next section is much more complicated however, as it involves calculating this invariant for our specific manifolds E_{ij}. To do so we will need to calculate Pontryagin classes explicitly, which is generally tricky to do.

3.3.4 Calculating $\lambda(E_{ij})$

Our strategy to calculate $\lambda(E_{ij})$ is as follows. We recall that the structural group of our topological spheres E_{ij} is $SO(4)$, and there is a one-to-one correspondence between these bundles and the elements of $\pi_3(SO(4)) \cong \mathbb{Z} \times \mathbb{Z}$. Using this fact, we calculate that the first Pontryagin class of E_{ij} is linear in i and j. Several observations will then lead us to our desired result:

Theorem 3.3.16. For $(i-j)^2 - 1 \not\equiv 0 \mod 7$, $(i+j=1)$ the manifold E_{ij} is homeomorphic but not diffeomorphic to S^7.

This section is loosely based on the work of McEnroe [McE15] and Braddell [Bra16].

We have already introduced the facts that the special orthogonal group $SO(4)$ is a topological group, and that there exists a unique (up to homotopy) G-principal bundle with the total space homotopic to a point. We can then consider the unique $SO(4)$-principal bundle $SO(4) \longrightarrow ESO(4) \longrightarrow BSO(4)$. But given any fibration $F \longrightarrow E \longrightarrow B$, there exists a long exact sequence of homotopy groups

$$... \to \pi_i(F) \to \pi_i(E) \to \pi_i(B) \to \pi_{i-1}(F) \to ...$$

In the case of a G-principal bundle, since the total space is contractible, $\pi_i(EG) = 0$, and so this gives us an isomorphism

$$\pi_i(BG) \cong \pi_{i-1}(G)$$

Corollary 3.3.17. $\pi_4(BSO(4)) \cong \pi_3(SO(4)) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Proof. The first isomorphism follows from the previous proposition. The second isomorphism is proven in Proposition 3.1.5. $\square$

We can now calculate the Pontryagin classes of our E_{ij}. We will have to take a somewhat indirect path to this, first considering the real, rank-4 vector bundles over S^4 we used to construct these sphere bundles and their Pontryagin classes. This section will mirror our initial construction of E_{ij}, and so hopefully will not seem too unfamiliar.

Proposition 3.3.18. The equivalence classes of real, rank-4 vector bundles over the 4-sphere with structural group $SO(4)$ are in one-to-one correspondence with elements of the group $\pi_3(SO(4)) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Proof. It suffices to specify clutching map $f_{ij} : S^3 \to SO(4)$ which is uniquely determined by $(i, j) \in \mathbb{Z} \oplus \mathbb{Z}$. But we have already done this in our construction of candidate exotic spheres. We defined $f_{ij} : S^3 \to SO(4)$ by its action using quaternionic multiplication: $f_{ij}(u) \cdot v := u^i v u^j$ for $\mathbb{R}^4$. This is indeed an isomorphism by our previous discussion, and so by definition each pair $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ gives rise to a unique rank-4 real vector bundle over S^4 denoted ξ_{ij}. $\square$

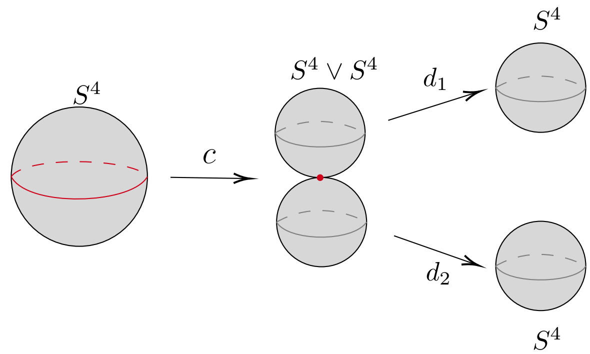
The next result is critical in finding an explicit equation for the invariant of ξ_{ij}.

Proposition 3.3.19. Denote the standard generator $\alpha \in H^4(S^4)$. Then

$$p_1(\xi_{ij}) = k(i - j)\alpha,$$

for some $k \in \mathbb{Z}$.

Proof. First, let us understand the group structure on $\pi_4(BSO(4))$. Consider the sequence of homotopy equivalences in the following figure.



where c is the collapse of the equator of S^4 (as in Figure 2.2.3), d_1 is the collapse of the bottom sphere and d_2 is the collapse of the bottom sphere.

The following map is then an isomorphism:

$$\phi : H^4(S^4) \times H^4(S^4) \to H^4(S^4 \vee S^4); \;\; \phi : (\alpha, \beta) \mapsto d_1^*(\alpha) + d_2^*(\beta). \tag{3.5}$$

Combining these maps, we have:

$$c^* \circ \phi : H^4(S^4) \times H^4(S^4) \to H^4(S^4); \;\; c^* \circ \phi : (\alpha, \beta) \mapsto \alpha + \beta. \tag{3.6}$$

Now suppose $[f], [g] \in \pi_4(BSO(4))$. Using the homotopy group definition using spheres, we consider f and g as maps $S^4 \to BSO(4)$. The group composition operation is given by

$$[f] + [g] := [(f \vee g) \circ c].$$

Define a map $\Theta : \pi_4(BSO(4)) \to H^4(S^4)$ by $\Theta : [f] \mapsto f^*(p_1(ESO(4))$. Using Equations 3.5 and 3.6, we can see that this is actually a group homomorphism:

$$\begin{aligned}
\Theta([f] + [g]) &= \Theta([(f \vee g) \circ c]) \\
&= (c^* \circ (f \vee g)^*)(p_1(ESO(4))) \\
&= (c^* \circ \phi) \circ (\phi^{-1} \circ (f \vee g)^*)(p_1(ESO(4))) \\
&= (c^* \circ \phi)(f^*(p_1(ESO(4))), g^*(p_1(ESO(4)))) \\
&= f^*(p_1(ESO(4))) + g^*(p_1(ESO(4))) \\
&= \Theta([f]) + \Theta([g]).
\end{aligned}$$

Recalling our one-to-one correspondence in Proposition 3.3.18 and the naturality property of Pontryagin classes in Proposition 2.5.12, we have then that the first Pontryagin class of ξ_{ij} is linear in i and j

$$p_1(\xi_{ij}) = (ai + bj)\alpha,$$

where α is a generator in $H^4(S^4) \cong \mathbb{Z}$ and $a, b \in \mathbb{Z}$.

Notice that in a natural way we have an isomorphism that reverses the orientation of real vector bundles by taking the quaternionic conjugate in each fibre [dM18]. The isomorphism sends $\xi_{ij} \to \xi_{-j-i}$. By the property that Pontryagin classes are invariant under change in orientation, this isomorphism implies that

$$p_1(\xi_{ij}) = (ai + bj)\alpha = (-aj - bi)\alpha \implies p_1(\xi_{ij}) = k(i - j)\alpha$$

for some $k \in \mathbb{Z}$. □

Our calculations need not stop here:

Proposition 3.3.20. The coefficient k in $p_1(\xi_{ij}) = k(i - j)\alpha$ is equal to ± 2.

Milnor's argument for proving this is only a couple of sentences long. However, as is the theme throughout this manuscript, proving this result is not easy as Milnor makes it seem. To do this, we will first take a detour through the world of quaternionic projective space, $\mathbb{HP}^n$. Some useful propositions and an isomorphism between E_{01} and a particular bundle of $\mathbb{HP}^n$ will allow us to derive the coefficient k.

First, what is the quaternionic projective plane? A reader familiar with the complex projective plane may find this section to be reminiscent of notions that they have formally encountered.

Definition 3.3.21. We define **quaternionic projective space** to be

$$\mathbb{HP}^n := \mathbb{H}^{n+1} \setminus \{0\} \big/ \mathbb{H}^*$$

Where $\mathbb{H}^*$ denotes the multiplicative group of nonzero quaternions, acting on $\mathbb{H}^{n+1} \setminus \{0\}$ as scalar multiplication on the left. This quotient group can then be identified as lines in quaternionic space $\mathbb{H}^{n+1}$ that pass through 0. This can be alternatively stated as

$$\mathbb{HP}^n := \mathbb{H}^{n+1} \setminus \{0\} \big/ \sim,$$

where $\sim$ is the equivalence relation identifying $u \sim \lambda u$ for $\lambda \in \mathbb{H} \setminus \{0\}$. We call $\mathbb{HP}^1$ the quaternionic projective line, and $\mathbb{HP}^2$ the quaternionic projective plane.

Remark 3.3.22. Notice that since quaternionic multiplication is non-commutative, we cannot define $\mathbb{HP}^n$ in exactly the same way that we would define $\mathbb{C}P^n$. That is, we must restrict our attention to either left or right multiplication. The convention is to take the left.

Proposition 3.3.23. $\mathbb{HP}^1$ is homeomorphic to S^4.

Proof. This is a well known result, and readers familiar with the fact that $\mathbb{C}P^1 \cong S^2$ can easily extend the intuitive reasoning behind it to the quaternionic case. Indeed, elements in $\mathbb{HP}^1$ can be represented by either $[v : 1]$ or $[1 : 0]$. Naturally the quaternionic projective line minus a point can be identified with the space of quaternions itself:

$$\mathbb{HP}^1 \setminus \{[1:0]\} \cong \mathbb{H} \cong \mathbb{R}^4.$$

We can extend the above homeomorphism to the one point compactification of $\mathbb{R}^4$, which is nothing but the 4-sphere, giving the desired result:

$$\mathbb{HP}^1 \cong S^4.$$

This can be formalised by the explicit map:

$$\phi : \mathbb{H}^2 \to S^4; \quad \phi : (u, v) \mapsto \left(\frac{2u\bar{v}}{\|u\|^2 + \|v\|^2}, \frac{\|u\|^2 - \|v\|^2}{\|u\|^2 + \|v\|^2} \right).$$

We observe that this is actually a homeomorphism $\mathbb{HP}^1 \to S^4$, noting that the map is invariant under scalar multiplication: $\phi(\lambda u, \lambda v) = \phi(u, v)$ for $\lambda \in \mathbb{H} \setminus \{0\}$. □

Definition 3.3.24. On $\mathbb{HP}^1$, we have a non-trivial line bundle known as the **tautological line bundle** given by the following:

$$\gamma := \{(l, v) \in \mathbb{HP}^1 \times \mathbb{H}^2 \;:\; [v] = l\}.$$

Proposition 3.3.25. We have an isomorphism $\xi_{01} \cong \gamma_{\mathbb{H}}$.

Proof. This proof is due to [Bra16] and [dM18]. Notice that by Proposition 3.3.23, ξ_{01} and $\gamma_{\mathbb{H}}$ are both real vector bundles over S^4. A trivialising atlas for $\gamma_{\mathbb{H}}$ is $\{(U_i, \psi_i)\}$, where $U_i := \{[u_0 : u_1] : u_i \neq 0\}$ and the trivialisations $\psi_i : U_i \to \mathbb{H}^2 \cong \mathbb{R}^8$ are given by

$$\psi_0 = ([1 : u_1], (t_0, t_0 u_1)) \mapsto (t_0, u_1)$$

and

$$\psi_1 = ([u_0 : 1], (t_1 u_0, t_1)) \mapsto (t_1, u_0).$$

Observe that

$$\psi_1 \circ \psi_0^{-1} : (t_0, u_1) \mapsto (t_0 u_1, u_1^{-1}).$$

Therefore the transition function of this bundle is then $U_0 \cap U_1 \to SL(4)$,

$$g([u_0, u_1]) \mapsto (v \mapsto v u_0^{-1} u_1).$$

But under the identification of $S^4 \cong \mathbb{HP}^1$, this is nothing but a scaled version of the transition functions of ξ_{01}:

$$\tilde{g}([u_0, u_1]) \mapsto \left(v \mapsto \frac{v u_0^{-1} u_1}{\|u_0^{-1} u_1\|} \right).$$

The isomorphism between ξ_{01} and $\gamma_{\mathbb{H}}$ is induced simply by scaling fibres.

$\square$

Theorem 3.3.26. The cohomology ring of the quaternionic projective line is generated by $e := e(\gamma_{\mathbb{H}})$ in the following way

$$H^*(\mathbb{HP}^1) \cong {}^{\mathbb{Z}[e]}\big/_{e^2}.$$

Proof. Denote by $\gamma_{\mathbb{H}}^*$ the total space of $\gamma_{\mathbb{H}}$ minus the zero section. Then $\gamma_{\mathbb{H}}^*$ is homotopic to $S^{4(1)+3} = S^7$. To see this, consider the map

$$\rho : \mathbb{H}^2 \setminus \{0\} \to \gamma_{\mathbb{H}}^*; \quad \rho : v \mapsto ([v], v).$$

This is clearly a homeomorphism. But $\mathbb{H}^2 \setminus \{0\}$ is trivially homotopic to S^7, which is exactly what we wanted to prove.

The above means that $H^k(\gamma_{\mathbb{H}}^*) \cong \mathbb{Z}$ for $i = 0, 7$, and zero otherwise. Dropping this into the Gysin sequence, we have the following LES

$$\ldots \longrightarrow H^{i-4}(\mathbb{H}P^1) \xrightarrow{\smile e} H^i(\mathbb{H}P^1) \xrightarrow{\pi^*} H^i(\gamma_{\mathbb{H}}^*) \longrightarrow H^{i-3}(\mathbb{H}P^1) \longrightarrow \ldots$$

where e denotes the Euler class $e \in H^4(\mathbb{HP}^1)$. Since $H^i(\mathbb{HP}^1) = 0$ trivially for $i < 0$, this gives us isomorphisms

$$H^i(\mathbb{HP}^1) \cong H^i(\gamma_{\mathbb{H}}^*) \quad \text{for } i < 3.$$

The final part of the sequence is more interesting:

$$0 \longrightarrow H^3(\mathbb{H}P^1) \xrightarrow{\pi^*} H^3(\gamma_{\mathbb{H}}^*) \longrightarrow H^0(\mathbb{H}P^1) \xrightarrow{\smile e} H^4(\mathbb{H}P^1) \xrightarrow{\pi^*} H^4(\gamma_{\mathbb{H}}^*) \longrightarrow \ldots$$

Combining this with the fact that $H^i(\gamma_{\mathbb{H}}^*) = 0$ for $i = 3, 4$, we have an isomorphism induced by cupping with the Euler class e:

$$H^0(\mathbb{H}P^1) \xrightarrow{\smile e} H^4(\mathbb{H}P^1)$$

The result then follows.

$\square$

Proposition 3.3.27. The Pontryagin class and Euler class of $\gamma_{\mathbb{H}}$ are related in the following way

$$p_1(\gamma_{\mathbb{H}}) = -2e(\gamma_{\mathbb{H}}).$$

Proof. We will first make a couple of observations. By definition, $e(\gamma_{\mathbb{H}}) \in H^4(\mathbb{HP}^1)$, and so the cohomology ring of $\mathbb{HP}^1$ contains only trivial elements of $H^2(\mathbb{HP}^1)$. This means that $c_1(\gamma_{\mathbb{H}}) = 0$. We also recall the relationship between the Pontryagin classes and total Chern class in Proposition 2.5.14:

$$(1 - p_1 + p_2 - ...) = (1 - c_1 + c_2 - ...)(1 + c_1 + c_2 + ...)$$

which in our case gives us

$$(1 - p_1) = (1 + c_2)(1 + c_2) = 1 + 2c_2 + c_2^2.$$

But we know that the top Chern class is equivalent to the Euler class, and $e^2 \equiv 0$, therefore

$$1 - p_1 = 1 + 2e \implies p_1 = -2e.$$

□

We have finally reached the point of being able to prove Proposition 3.3.20:

Proof. We have two expressions $p_1(\xi_{ij}) = k(i-j)\alpha$ and $p_1(\xi_{ij}) = p_1(\gamma_{\mathbb{H}}) = -2e(\gamma_{\mathbb{H}})$. But by Theorem 3.3.26, α is nothing but $e(\gamma_{\mathbb{H}})$ up to a choice of sign (orientation). It must then be that $k = \pm 2$. □

Recalling Definition 3.3.8, we are only halfway through the story: we have calculated $p_1(\xi_{ij})$, but we haven't calculated $p_1(N^8)$ for some 8-manifold with boundary E_{ij}. But since we know that λ is invariant under choice of N^8, we can simply consider the associated disc bundle D_{ij} which has E_{ij} as its boundary. Recall this bundle from Section 3.1, where we construted both D_{ij} and E_{ij} by sequential restrictions on the fibres of the vector bundle ξ_{ij}.

Theorem 3.3.28. $\lambda(E_{ij}) \equiv (i-j)^2 - 1 \mod 7$.

Proof. This argument follows [Bra16], although near identical ones are offered in [McE15] and [dM18]. For this section, it is important to note the alternation between the notation of $p_1(X)$ and $p_1(TX)$. These are equivalent in the same sense of Remark 3.3.7.

Notice that the inclusion and projection maps $D_{ij} \overset{i}{\hookrightarrow} \xi_{ij} \overset{\pi}{\to} S^4$ are actually homotopy equivalences, and induce isomorphisms between their cohomology groups. It follows then that the cohomology of $H^4(D_{ij})$ is generated by $\beta := i^*\pi^*(\alpha)$.

With the above established, fix an orientation on D_{ij} (and therefore E_{ij}) by choosing $[\nu] \in H^8(D_{ij}, E_{ij})$ such that $\langle (i^{-1}(\beta))^2, [\nu] \rangle = 1$. It follows then that $\sigma(D_{ij}, E_{ij}) = 1$. By our inclusion map, $TD_{ij} = T\xi_{ij}|_{D_{ij}}$. We can consider $T\xi_{ij}$ as the Whitney sum

$$T\xi_{ij} = \pi^*(\xi_{ij}) \oplus \pi^*(TS^4),$$

i.e. the Whitney sum of bundles tangent to and normal to the fibres of ξ_{ij}. Since $H^*(\xi_{ij})$ is torsion-free, by the whitney sum formula and the naturality property of Pontryagin classes,

$$\begin{aligned} p_1(T\xi_{ij}) =& p_1(\pi^*(\xi_{ij}) \oplus \pi^*(TS^4)) \\ =& p_1(\pi^*(\xi_{ij})) + p_1(\pi^*(TS^4)) \\ =& \pi^*(p_1(\xi_{ij})) + \pi^*(p_1(TS^4)) \\ =& \pi^*(\pm 2(i-j)\alpha) + \pi^*(0) \\ =& \pm 2(i-j)\pi^*(\alpha), \end{aligned}$$

where we used the fact that the first Pontryagin class $p_1(TS^4) = 0$ as discussed in Example 2.5.16. Pulling this back via i^*, it follows that

$$p_1(D_{ij}) = \pm 2(i-j)i^*\pi^*(\alpha) = \pm 2(i-j)\beta.$$

We can now give our explicit identity for $\lambda(E_{ij})$:

$$\begin{aligned} \lambda(E_{ij}) =& 2q(D_{ij}) - \sigma(D_{ij}) \mod 7 \\ =& 2q(D_{ij}) - 1 \mod 7 \\ =& 2\langle [\nu], i^{-1}(p_1(D_{ij}))^2 \rangle - 1 \mod 7 \\ =& 2\langle [\nu], i^{-1}(\pm 2(i-j)\beta)^2 \rangle - 1 \mod 7 \\ =& 8(i-j)^2 \langle [\nu], i^{-1}(\beta)^2 \rangle - 1 \mod 7 \\ =& 8(i-j)^2 - 1 \mod 7 \\ \equiv& (i-j)^2 - 1 \mod 7, \end{aligned}$$

proving the claim. □

Proposition 3.3.29. For $(i-j)^2 \not\equiv 1 \mod 7$, E_{ij} is an exotic sphere. That is, E_{ij} is not diffeomorphic to the standard 7-sphere.

Proof. Remark 3.3.14 says that if E_{ij} is diffeomorphic to S^7, then $\lambda(E_{ij}) = \lambda(S^7)$. But $\lambda(S^7) = 0$ (Corollary 3.3.15), and so

$$\implies (i-j)^2 - 1 \equiv 0 \mod 7.$$

If the above identity does not hold, then our assumption must be wrong: E_{ij} cannot be diffeomorphic to S^7. □

Remark 3.3.30. Considering all possible values of $(i-j)^2-1$, there are at least 4 differential structures on S^7.

Chapter 4

Further Research and Concluding Remarks

Milnor's proof justifying the existence of exotic spheres is truly remarkable. For his work, Milnor was awarded several distinctions, most prominently the Fields Medal in 1962 [IMU62], and the Abel Prize in 2011 [Mil11a]. The existence of exotic structures forged a distinction between the fields of differential geometry and topology, disproving the differentiable Poincaré hypothesis in general. The discovery also led to greater interest in the field of differential topology, which was still in its infancy in the mid 20$^{\text{th}}$ century.

Milnor's discovery also sparked a hunt for explicit exotic structures in other dimensions. In 1966, Egbert Brieskorn discovered what are known as *Brieskorn spheres*, manifolds constructed by intersecting complex spheres with certain complex hypersurfaces. Brieskorn's discovery classified all 28 possible smooth structures on the oriented 7-sphere, as well as all other exotic spheres of dimension $4m-1$ for integers $m > 1$ [Bri66].

Milnor's own subsequent work focused on more general classifications. In 1959, Milnor proved that the set of differentiable structures on the n-sphere form a monoid. Later in 1963, Kervaire and Milnor [KM63] showed that this monoid, away from $n = 4$, is in fact an abelian group, isomorphic to the groups of *h-cobordism classes* $[\Theta_n]$ of oriented homotopy n-spheres. They were able to explicitly calculate the orders of these groups for $1 \leq n \leq 18$, the results of which are summarised in Table 4.1, adapted from [Mil11b].

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$[\Theta_n]$	1	1	1	?	1	1	28	2	8	6	992	1	3	2	16256	2	16	16

Table 4.1: Order of group of differentiable structures on S^n.

The case of $n = 4$ remains an open question, and is actively being researched today. Though little is understood, what we do know is that either there are no exotic structures on S^4, or infinitely many. The discovery of a single exotic 4-sphere prove the latter to be true. Research on exotic structures for other manifolds alludes to the mystery of dimension 4. In the case of Euclidean space, R^n has no exotic smooth structures for all $\mathbb{N} \setminus \{4\}$, whereas for $n = 4$, there are uncountably many [F$^+$82], [Don83], [Tau87].

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